

# SINGULAR QUADRATIC LIE SUPERALGEBRAS

MINH THANH DUONG, ROSANE USHIROBIRA

**ABSTRACT.** In this paper, we give a generalization of results in [PU07] and [DPU] by applying the tools of graded Lie algebras to quadratic Lie superalgebras. In this way, we obtain a numerical invariant of quadratic Lie superalgebras and a classification of singular quadratic Lie superalgebras, i.e. those with a nonzero invariant. Finally, we study a class of quadratic Lie superalgebras obtained by the method of generalized double extensions.

## 0. INTRODUCTION

Throughout the paper, the base field is  $\mathbb{C}$  and all vector spaces are complex and finite-dimensional. We denote the ring  $\mathbb{Z}/2\mathbb{Z}$  by  $\mathbb{Z}_2$  as in superalgebra theory.

Let us begin with a  $\mathbb{Z}_2$ -graded vector space  $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ . We denote by  $\text{Alt}(\mathfrak{g}_{\bar{0}})$  the Grassmann algebra of  $\mathfrak{g}_{\bar{0}}$ , that is, the algebra of alternating multilinear forms on  $\mathfrak{g}_{\bar{0}}$  equipped with the wedge product and by  $\text{Sym}(\mathfrak{g}_{\bar{1}})$  the algebra of symmetric multilinear forms on  $\mathfrak{g}_{\bar{1}}$ .

We say that  $\mathfrak{g}$  is a *quadratic*  $\mathbb{Z}_2$ -graded vector space if it is endowed with a non-degenerate even supersymmetric bilinear form  $B$  (that is,  $B$  is symmetric on  $\mathfrak{g}_{\bar{0}}$ , skew-symmetric on  $\mathfrak{g}_{\bar{1}}$  and  $B(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}) = 0$ ). In addition, if there is a Lie superalgebra structure  $[\cdot, \cdot]$  on  $\mathfrak{g}$  such that  $B$  is invariant, i.e.  $B([X, Y], Z) = B(X, [Y, Z])$  for all  $X, Y, Z \in \mathfrak{g}$ , then  $\mathfrak{g}$  is called a *quadratic* (or *orthogonal* or *metrised*) Lie superalgebra.

Algebras endowed with an invariant bilinear form appear in many areas of Mathematics and Physics and they are a remarkable algebraic object. A structural theory of quadratic Lie algebras, based on the notion of a double extension (a combination of a central extension and a semi-direct product), was introduced by V. Kac [Kac85] in the solvable case and by A. Medina and P. Revoy [MR85] in the general case. Another interesting construction, the  $T^*$ -extension, based on the notion of a generalized semi-direct product of a Lie algebra and its dual space was given by M. Bordemann [Bor97] for solvable quadratic Lie algebras. Both notions have been generalized for quadratic Lie superalgebras in papers by H. Benamor and S. Benayadi [BB99] and by I. Bajo, S. Benayadi and M. Bordemann [BBB].

A third approach, based on the concept of super Poisson bracket, was introduced in [PU07], providing several interesting properties of quadratic Lie algebras: the

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authors consider  $(\mathfrak{g}, [\cdot, \cdot], B)$  a non-Abelian quadratic Lie algebra and define a 3-form  $I$  on  $\mathfrak{g}$  by

$$I(X, Y, Z) = B([X, Y], Z), \forall X, Y, Z \in \mathfrak{g}.$$

Then  $I$  is nonzero and  $\{I, I\} = 0$ , where  $\{\cdot, \cdot\}$  is the super Poisson bracket defined on the  $(\mathbb{Z}$ -graded) Grassmann algebra  $\text{Alt}(\mathfrak{g})$  of  $\mathfrak{g}$ , by

$$\{\Omega, \Omega'\} = (-1)^{\deg_{\mathbb{Z}}(\Omega)+1} \sum_{j=1}^n \iota_{X_j}(\Omega) \wedge \iota_{X_j}(\Omega'), \forall \Omega, \Omega' \in \text{Alt}(\mathfrak{g})$$

with  $\{X_1, \dots, X_n\}$  a fixed orthonormal basis of  $\mathfrak{g}$ .

Conversely, given a quadratic vector space  $(\mathfrak{g}, B)$  and a nonzero 3-form  $I \in \text{Alt}^3(\mathfrak{g})$  satisfying  $\{I, I\} = 0$ , then there is a non-Abelian Lie algebra structure on  $\mathfrak{g}$  such that  $B$  is invariant.

The element  $I$  carries some useful information about corresponding quadratic Lie algebras. For instance, when  $I$  is decomposable and nonzero, corresponding quadratic Lie algebras are called *elementary* quadratic Lie algebras and they are exhaustively classified in [PU07]. This classification is based on basic properties of quadratic forms and the super Poisson bracket. In this case,  $\dim([\mathfrak{g}, \mathfrak{g}]) = 3$  and coadjoint orbits have dimension at most 2. In [DPU], the authors consider further a notion that is called the *dup-number* of a non-Abelian quadratic Lie algebra  $\mathfrak{g}$ . It is defined by  $\text{dup}(\mathfrak{g}) = \dim(\{\alpha \in \mathfrak{g}^* \mid \alpha \wedge I = 0\})$  where  $\mathfrak{g}^*$  is the dual space of  $\mathfrak{g}$ . The dup-number receives values 0, 1 or 3 and it measures the decomposability of  $I$ . For instance,  $I$  is decomposable if and only if  $\text{dup}(\mathfrak{g}) = 3$ . Moreover, it is also a numerical invariant of quadratic Lie algebras under Lie algebra isomorphisms, meaning that if  $\mathfrak{g}$  and  $\mathfrak{g}'$  are isomorphic quadratic Lie algebras then  $\text{dup}(\mathfrak{g}) = \text{dup}(\mathfrak{g}')$ . Its proof is rather non-trivial. It is obtained through a description of the space generated by invariant symmetric bilinear forms on a quadratic Lie algebra with nonzero dup-number. Such quadratic Lie algebra is called a *singular*. An unexpected property is that there are many non-degenerate invariant symmetric bilinear forms on a singular quadratic Lie algebra. Though they can be linearly independent all of them are equivalent in the solvable case, i.e. two solvable singular quadratic Lie algebras with same Lie algebra structure are isometrically isomorphic (or *i-isomorphic*, for short). In other words, isomorphic and *i-isomorphic* notions are equivalent on solvable singular quadratic Lie algebras. Another remarkable result is that all singular quadratic Lie algebras are classified up to isomorphism by  $O(n)$ -adjoint orbits of the Lie algebra  $\mathfrak{o}(n)$ .

The purpose of this paper is to give a interpretation of this last approach for quadratic Lie superalgebras. We combine it with the notion of double extension as it was done for quadratic Lie algebras [DPU]. Further, we use the notion of generalized double extension. In result, we obtain a rather colorful picture of quadratic Lie superalgebras.

Let us give some details of our main results. First, let  $\mathfrak{g}$  be a quadratic  $\mathbb{Z}_2$ -graded vector space. Recall that  $\text{Alt}(\mathfrak{g}_{\overline{0}})$  and  $\text{Sym}(\mathfrak{g}_{\overline{1}})$  are  $\mathbb{Z}$ -graded algebras and this gradation can be used to define a  $\mathbb{Z} \times \mathbb{Z}_2$ -gradation on each algebra. Consider

then the  $\mathbb{Z} \times \mathbb{Z}_2$ -graded *super-exterior algebra* of  $\mathfrak{g}^*$  defined by

$$\mathcal{E}(\mathfrak{g}) = \text{Alt}(\mathfrak{g}_{\bar{0}}) \underset{\mathbb{Z} \times \mathbb{Z}_2}{\otimes} \text{Sym}(\mathfrak{g}_{\bar{1}})$$

with the natural *super-exterior product*

$$(\Omega \otimes F) \wedge (\Omega' \otimes F') = (-1)^{\deg_{\mathbb{Z}}(F) \deg_{\mathbb{Z}}(\Omega')} (\Omega \wedge \Omega') \otimes FF',$$

for all  $\Omega, \Omega'$  in  $\text{Alt}(\mathfrak{g}_{\bar{0}})$  and  $F, F'$  in  $\text{Sym}(\mathfrak{g}_{\bar{1}})$ . It is clear that  $\mathcal{E}$  is commutative and associative. For more details of the algebra  $\mathcal{E}(\mathfrak{g})$  the reader should refer to [Sch79], [BP89] or [MPU09].

In [MPU09], the authors use the *super  $\mathbb{Z} \times \mathbb{Z}_2$ -Poisson bracket*  $\{\cdot, \cdot\}$  on the super-exterior algebra  $\mathcal{E}(\mathfrak{g})$  defined as follows:

$$\{\Omega \otimes F, \Omega' \otimes F'\} = (-1)^{\deg_{\mathbb{Z}}(F) \deg_{\mathbb{Z}}(\Omega')} (\{\Omega, \Omega'\} \otimes FF' + (\Omega \wedge \Omega') \otimes \{F, F'\}),$$

for all  $\Omega, \Omega' \in \text{Alt}(\mathfrak{g}_{\bar{0}})$ ,  $F, F' \in \text{Sym}(\mathfrak{g}_{\bar{1}})$ .

In Section 1, we will recall some simple properties of the super  $\mathbb{Z} \times \mathbb{Z}_2$ -Poisson bracket that are necessary for our purpose.

It is easy to check that for a quadratic Lie superalgebra  $(\mathfrak{g}, B)$ , if we define a trilinear form  $I$  on  $\mathfrak{g}$  by

$$I(X, Y, Z) = B([X, Y], Z), \quad \forall X, Y, Z \in \mathfrak{g}$$

then  $I \in \mathcal{E}^{(3, \bar{0})}(\mathfrak{g})$ . Therefore, it seems to be natural to ask: when does  $\{I, I\} = 0$ ? We shall give an affirmative answer to this in Proposition 1.11. Moreover, similarly to the Lie algebra case, we show that non-Abelian quadratic Lie superalgebra structures on a quadratic  $\mathbb{Z}_2$ -graded vector space  $(\mathfrak{g}, B)$  are in one-to-one correspondence with nonzero elements  $I$  in  $\mathcal{E}^{(3, \bar{0})}(\mathfrak{g})$  satisfying  $\{I, I\} = 0$ .

In Section 2, we introduce the notion of *dup-number* for a non-Abelian quadratic Lie superalgebra  $\mathfrak{g}$ :

$$\text{dup}(\mathfrak{g}) = \dim(\{\alpha \in \mathfrak{g}^* \mid \alpha \wedge I = 0\})$$

and consider the set of quadratic Lie superalgebras with nonzero dup-number: the set of singular quadratic Lie superalgebras. Similarly to quadratic Lie algebras, if  $\mathfrak{g}$  is non-Abelian then  $\text{dup}(\mathfrak{g}) \in \{0, 1, 3\}$ . Thanks to Lemma 2.1, if  $\text{dup}(\mathfrak{g}) = 3$  then  $\mathfrak{g}_{\bar{1}}$  is a central ideal of  $\mathfrak{g}$  and  $\mathfrak{g}_{\bar{0}}$  is an elementary quadratic Lie algebra. Therefore, we focus on singular quadratic Lie superalgebras  $\mathfrak{g}$  with  $\text{dup}(\mathfrak{g}) = 1$ . We call them *singular quadratic Lie superalgebras of type  $S_1$* . However, differently than the Lie algebra case, the element  $I$  may be decomposable.

We list in Section 3 all non-Abelian reduced quadratic Lie superalgebras with  $I$  decomposable (see Definition 2.5 for the definition of a reduced quadratic Lie superalgebra). We call them *elementary quadratic Lie superalgebras*. In this case,  $\text{dup}(\mathfrak{g})$  is nonzero. In particular, if  $\text{dup}(\mathfrak{g}) = 3$  then  $\mathfrak{g}$  is an elementary quadratic Lie algebra. If  $\text{dup}(\mathfrak{g}) = 1$  then we obtain three quadratic Lie superalgebras with 2-dimensional even part. Actually, we prove in Proposition 4.1 that if  $\mathfrak{g}$  is a non-Abelian quadratic Lie superalgebra with 2-dimensional even part then  $\text{dup}(\mathfrak{g}) = 1$ .

Section 4 details a study of quadratic Lie superalgebras with 2-dimensional even part. We apply the concept of double extension as in [DPU] with a little change

by replacing a quadratic vector space by a symplectic vector space and keeping the other conditions (see Definition 4.6). Then we obtain a structure that we still call a *double extension* and one has (Proposition 4.8):

THEOREM 1:

*A quadratic Lie superalgebra has a 2-dimensional even part if and only if it is a double extension.*

By a very similar process as in [DPU] for solvable singular quadratic Lie algebras, a classification of quadratic Lie superalgebras with 2-dimensional even part up to isomorphism is given as follows. Let  $\mathcal{S}(2+2n)$  be the set of such structures on  $\mathbb{C}^{2+2n}$ . We call an algebra  $\mathfrak{g} \in \mathcal{S}(2+2n)$  *diagonalizable* (resp. *invertible*) if it is the double extension by a diagonalizable (resp. invertible) map. Denote the subsets of nilpotent elements, diagonalizable elements and invertible elements in  $\mathcal{S}(2+2n)$ , respectively by  $\mathcal{N}(2+2n)$ ,  $\mathcal{D}(2+2n)$  and by  $\mathcal{S}_{\text{inv}}(2+2n)$ . Denote by  $\widehat{\mathcal{N}}(2+2n)$ ,  $\widehat{\mathcal{D}}(2+2n)$ ,  $\widehat{\mathcal{S}_{\text{inv}}}(2+2n)$  the sets of isomorphic classes in  $\mathcal{N}(2+2n)$ ,  $\mathcal{D}(2+2n)$ ,  $\mathcal{S}_{\text{inv}}(2+2n)$  respectively and by  $\widehat{\mathcal{D}}_{\text{red}}(2+2n)$  the subset of  $\widehat{\mathcal{D}}(2+2n)$  including reduced ones. Also, we denote by  $\mathbb{P}^1(\mathfrak{sp}(2n))$  the projective space of  $\mathfrak{sp}(2n)$  with the action induced by the  $\text{Sp}(2n)$ -adjoint action on  $\mathfrak{sp}(2n)$ . Then we have the classification result (Propositions 4.13 and 4.14 and Appendix):

THEOREM 2:

- (i) *Let  $\mathfrak{g}$  and  $\mathfrak{g}'$  be elements in  $\mathcal{S}(2+2n)$ . Then  $\mathfrak{g}$  and  $\mathfrak{g}'$  are isometrically isomorphic and if and only if they are isomorphic.*
- (ii) *There is a bijection between  $\widehat{\mathcal{N}}(2+2n)$  and the set of nilpotent  $\text{Sp}(2n)$ -adjoint orbits of  $\mathfrak{sp}(2n)$ . It induces a bijection between  $\widehat{\mathcal{N}}(2+2n)$  and the set of partitions  $\mathcal{P}_{-1}(2n)$  of  $2n$  in which odd parts occur with even multiplicity.*
- (iii) *There is a bijection between  $\widehat{\mathcal{D}}(2+2n)$  and the set of semisimple  $\text{Sp}(2n)$ -orbits of  $\mathbb{P}^1(\mathfrak{sp}(2n))$ .*
- (iv) *There is a bijection between  $\widehat{\mathcal{S}_{\text{inv}}}(2+2n)$  and the set of invertible  $\text{Sp}(2n)$ -orbits of  $\mathbb{P}^1(\mathfrak{sp}(2n))$ .*
- (v) *There is a bijection between  $\widehat{\mathcal{S}}(2+2n)$  and the set of  $\text{Sp}(2n)$ -orbits of  $\mathbb{P}^1(\mathfrak{sp}(2n))$ .*

As for quadratic Lie algebras, we have the notion of quadratic dimension of a quadratic Lie superalgebra. In the case  $\mathfrak{g}$  is a quadratic Lie superalgebra having a 2-dimensional even part, we can compute its quadratic dimension as follows:

$$d_q(\mathfrak{g}) = 2 + \frac{(\dim(\mathcal{Z}(\mathfrak{g}) - 1))(\dim(\mathcal{Z}(\mathfrak{g}) - 2))}{2}.$$

where  $\mathcal{Z}(\mathfrak{g})$  is the center of  $\mathfrak{g}$ . It indicates that there are many non-degenerate invariant even supersymmetric bilinear forms on a quadratic Lie superalgebra with 2-dimensional even part but by Theorem 2 (i), all of them are equivalent.

Section 5 contains more results on a singular quadratic Lie superalgebra  $(\mathfrak{g}, B)$  of type  $S_1$ , that is, those with 1-valued dup-number. The first result is that  $\mathfrak{g}_{\bar{0}}$  is solvable and so  $\mathfrak{g}$  is solvable. Moreover, by Definition 5.3 and Lemma 5.5, the Lie superalgebra  $\mathfrak{g}$  can be realized as the double extension of a quadratic  $\mathbb{Z}_2$ -graded vector space  $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$  by a map  $\bar{C} = \bar{C}_0 + \bar{C}_1 \in \mathfrak{o}(\mathfrak{q}_{\bar{0}}) \oplus \mathfrak{sp}(\mathfrak{q}_{\bar{1}})$ . Denote by  $\mathcal{L}(\mathfrak{q}_{\bar{0}})$  (resp.  $\mathcal{L}(\mathfrak{q}_{\bar{1}})$ ) the set of endomorphisms of  $\mathfrak{q}_{\bar{0}}$  (resp.  $\mathfrak{q}_{\bar{1}}$ ). We give isomorphic and i-isomorphic characterizations of two singular quadratic Lie superalgebras of type  $S_1$  as follows (Proposition 5.7).

**THEOREM 3:**

*Let  $\mathfrak{g}$  and  $\mathfrak{g}'$  be two double extensions of  $\mathfrak{q}$  by  $\bar{C} = \bar{C}_0 + \bar{C}_1$  and  $\bar{C}' = \bar{C}'_0 + \bar{C}'_1$ , respectively. Assume that  $\bar{C}_1$  is nonzero. Then*

- (1) *there exists a Lie superalgebra isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}'$  if and only if there exist invertible maps  $P \in \mathcal{L}(\mathfrak{q}_{\bar{0}})$ ,  $Q \in \mathcal{L}(\mathfrak{q}_{\bar{1}})$  and a nonzero  $\lambda \in \mathbb{C}$  such that*
  - (i)  $\bar{C}'_0 = \lambda P \bar{C}_0 P^{-1}$  and  $P^* P \bar{C}_0 = \bar{C}_0$ .
  - (ii)  $\bar{C}'_1 = \lambda Q \bar{C}_1 Q^{-1}$  and  $Q^* Q \bar{C}_1 = \bar{C}_1$ .*where  $P^*$  and  $Q^*$  are the adjoint maps of  $P$  and  $Q$  with respect to  $B|_{\mathfrak{q}_{\bar{0}} \times \mathfrak{q}_{\bar{0}}}$  and  $B|_{\mathfrak{q}_{\bar{1}} \times \mathfrak{q}_{\bar{1}}}$ .*
- (2) *there exists an i-isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}'$  if and only if there is a nonzero  $\lambda \in \mathbb{C}$  such that  $\bar{C}'_0$  is in the  $O(\mathfrak{q}_{\bar{0}})$ -adjoint orbit through  $\lambda \bar{C}_0$  and  $\bar{C}'_1$  is in the  $Sp(\mathfrak{q}_{\bar{1}})$ -adjoint orbit through  $\lambda \bar{C}_1$ .*

We recall a remarkable result in [DPU] that two solvable singular quadratic Lie algebras are i-isomorphic if and only if they are isomorphic. A similar situation occurs for two quadratic Lie superalgebras with 2-dimensional even part as in Theorem 2. Therefore, there is a very natural question: is this result also true for two singular quadratic Lie superalgebras? We have an affirmative answer as follows (Proposition 5.15 for  $\text{dup}(\mathfrak{g}) = 1$  and [DPU] for  $\text{dup}(\mathfrak{g}) = 3$ ):

**THEOREM 4:**

*Let  $\mathfrak{g}$  and  $\mathfrak{g}'$  be two solvable singular quadratic Lie superalgebras. Then  $\mathfrak{g}$  and  $\mathfrak{g}'$  are i-isomorphic if and only if they are isomorphic.*

We close the problem on singular quadratic Lie superalgebras with an assertion that (Proposition 5.16):

**THEOREM 5:**

*The dup-number is invariant under Lie superalgebra isomorphism.*

As a consequence of its proof, we obtain a formula for the quadratic dimension of reduced singular quadratic Lie superalgebras of type  $S_1$  having  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$ .

In the last Section, we study the structure of a quadratic Lie superalgebra  $\mathfrak{g}$  such that its element  $I$  has the form:

$$I = J \wedge p$$

where  $p \in \mathfrak{g}_\tau^*$  is nonzero and  $J \in \text{Alt}^1(\mathfrak{g}_\tau) \otimes \text{Sym}^1(\mathfrak{g}_\tau)$  is indecomposable. We call  $\mathfrak{g}$  a *quasi-singular quadratic Lie superalgebra*. With the notion of *generalized double extension* given by I. Bajo, S. Benayadi and M. Bordemann in [BBB], we prove that (Corollary 6.5 and Proposition 6.9):

**THEOREM 6:**

*A quasi-singular quadratic Lie superalgebra is a generalized double extension of a quadratic  $\mathbb{Z}_2$ -graded vector space. This superalgebra is 2-nilpotent.*

In the Appendix, we recall fundamental results in the classification of  $O(m)$ -adjoint orbits of  $\mathfrak{o}(m)$  and  $\text{Sp}(2n)$ -adjoint orbits of  $\mathfrak{sp}(2n)$ . The classification of nilpotent and semisimple orbits is well-known. We further give here the classification of *invertible* orbits, i.e. orbits of isomorphisms in  $\mathfrak{o}(m)$  and  $\mathfrak{sp}(2n)$ . By the Fitting decomposition, we obtain a complete classification in the general case.

Many concepts used in this paper are generalizations of the quadratic Lie algebra case. We do not recall their original definitions here. For more details the reader can refer to [PU07] and [DPU].

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This article is dedicated to our admirable mentor Georges Pinczon (1948 – 2010). He suggested the main idea in Section 1 and discussed results in Sections 2 and 3.

## 1. APPLICATIONS OF GRADED LIE ALGEBRAS TO QUADRATIC LIE SUPERALGEBRAS

Let  $\mathfrak{g} = \mathfrak{g}_\tau \oplus \mathfrak{g}_\tau^*$  be a  $\mathbb{Z}_2$ -graded vector space. We call  $\mathfrak{g}_\tau$  and  $\mathfrak{g}_\tau^*$  respectively the *even* and the *odd* part of  $\mathfrak{g}$ . We begin by reviewing the construction of the super-exterior algebra of the dual space  $\mathfrak{g}^*$  of  $\mathfrak{g}$ . Then we define the super  $\mathbb{Z} \times \mathbb{Z}_2$ -Poisson bracket on  $\mathfrak{g}^*$  (for more details, see [MPU09] and [Sch79]).

**1.1. The super-exterior algebra of  $\mathfrak{g}^*$ .** Denote by  $\text{Alt}(\mathfrak{g}_\tau)$  the algebra of alternating multilinear forms on  $\mathfrak{g}_\tau$  and by  $\text{Sym}(\mathfrak{g}_\tau)$  the algebra of symmetric multilinear forms on  $\mathfrak{g}_\tau$ . Recall that  $\text{Alt}(\mathfrak{g}_\tau)$  is the exterior algebra of  $\mathfrak{g}_\tau^*$  and  $\text{Sym}(\mathfrak{g}_\tau)$  is the symmetric algebra of  $\mathfrak{g}_\tau^*$ . These algebras are  $\mathbb{Z}$ -graded algebras. We define a  $\mathbb{Z} \times \mathbb{Z}_2$ -gradation on  $\text{Alt}(\mathfrak{g}_\tau)$  and on  $\text{Sym}(\mathfrak{g}_\tau)$  by

$$\text{Alt}^{(i, \bar{0})}(\mathfrak{g}_\tau) = \text{Alt}^i(\mathfrak{g}_\tau), \quad \text{Alt}^{(i, \bar{1})}(\mathfrak{g}_\tau) = \{0\}$$

$$\text{and } \text{Sym}^{(i, \bar{i})}(\mathfrak{g}_\tau) = \text{Sym}^i(\mathfrak{g}_\tau), \quad \text{Sym}^{(i, \bar{j})}(\mathfrak{g}_\tau) = \{0\} \quad \text{if } \bar{i} \neq \bar{j},$$

where  $i, j \in \mathbb{Z}$  and  $\bar{i}, \bar{j}$  are respectively the residue classes modulo 2 of  $i$  and  $j$ .

The *super-exterior algebra* of  $\mathfrak{g}^*$  is the  $\mathbb{Z} \times \mathbb{Z}_2$ -graded algebra defined by:

$$\mathcal{E}(\mathfrak{g}) = \text{Alt}(\mathfrak{g}_\tau) \underset{\mathbb{Z} \times \mathbb{Z}_2}{\otimes} \text{Sym}(\mathfrak{g}_\tau)$$



endowed with the *super-exterior product* on  $\mathcal{E}(\mathfrak{g})$ :

$$(\Omega \otimes F) \wedge (\Omega' \otimes F') = (-1)^{f\omega'} (\Omega \wedge \Omega') \otimes FF',$$

for all  $\Omega \in \text{Alt}(\mathfrak{g}_{\bar{0}})$ ,  $\Omega' \in \text{Alt}^{\omega'}(\mathfrak{g}_{\bar{0}})$ ,  $F \in \text{Sym}^f(\mathfrak{g}_{\bar{1}})$ ,  $F' \in \text{Sym}(\mathfrak{g}_{\bar{1}})$ . Remark that the  $\mathbb{Z} \times \mathbb{Z}_2$ -gradation on  $\mathcal{E}(\mathfrak{g})$  is given by:

$$\text{if } A = \Omega \otimes F \in \text{Alt}^{\omega}(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^f(\mathfrak{g}_{\bar{1}}) \text{ with } \omega, f \in \mathbb{Z}, \text{ then } A \in \mathcal{E}^{(\omega+f, \bar{f})}(\mathfrak{g}).$$

So, in terms of the  $\mathbb{Z}$ -gradations of  $\text{Alt}(\mathfrak{g}_{\bar{0}})$  and  $\text{Sym}(\mathfrak{g}_{\bar{1}})$ , we have:

$$\mathcal{E}^n(\mathfrak{g}) = \bigoplus_{m=0}^n (\text{Alt}^m(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^{n-m}(\mathfrak{g}_{\bar{1}}))$$

and in terms of the  $\mathbb{Z}_2$ -gradations,

$$\mathcal{E}_{\bar{0}}(\mathfrak{g}) = \text{Alt}(\mathfrak{g}_{\bar{0}}) \otimes \left( \bigoplus_{j \geq 0} \text{Sym}^{2j}(\mathfrak{g}_{\bar{1}}) \right) \text{ and } \mathcal{E}_{\bar{1}}(\mathfrak{g}) = \text{Alt}(\mathfrak{g}_{\bar{0}}) \otimes \left( \bigoplus_{j \geq 0} \text{Sym}^{2j+1}(\mathfrak{g}_{\bar{1}}) \right).$$

Notice that the graded vector space  $\mathcal{E}(\mathfrak{g})$  endowed with this product is a commutative and associative graded algebra.

Another equivalent construction is given in [BP89]:  $\mathcal{E}(\mathfrak{g})$  is the graded algebra of super-antisymmetric multilinear forms on  $\mathfrak{g}$ . The algebras  $\text{Alt}(\mathfrak{g}_{\bar{0}})$  and  $\text{Sym}(\mathfrak{g}_{\bar{1}})$  are regarded as subalgebras of  $\mathcal{E}(\mathfrak{g})$  by identifying  $\Omega := \Omega \otimes 1$ ,  $F := 1 \otimes F$ , and the tensor product  $\Omega \otimes F = (\Omega \otimes 1) \wedge (1 \otimes F)$  for all  $\Omega \in \text{Alt}(\mathfrak{g}_{\bar{0}})$ ,  $F \in \text{Sym}(\mathfrak{g}_{\bar{1}})$ .

**1.2. The super  $\mathbb{Z} \times \mathbb{Z}_2$ -Poisson bracket on  $\mathcal{E}(\mathfrak{g}^*)$ .** Let us assume that the vector space  $\mathfrak{g}$  is equipped with a non-degenerate even supersymmetric bilinear form  $B$ . That means

$$B(X, Y) = (-1)^{xy} B(Y, X)$$

for all homogeneous  $X \in \mathfrak{g}_x$ ,  $Y \in \mathfrak{g}_y$  and  $B(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}) = 0$ . In this case,  $\dim(\mathfrak{g}_{\bar{1}})$  must be even and  $\mathfrak{g}$  is also called a *quadratic  $\mathbb{Z}_2$ -graded vector space*.

The Poisson bracket on  $\text{Sym}(\mathfrak{g}_{\bar{1}})$  and the super Poisson bracket on  $\text{Alt}(\mathfrak{g}_{\bar{0}})$  are defined as follows. Let  $\mathcal{B} = \{X_1, \dots, X_n, Y_1, \dots, Y_n\}$  be a Darboux basis of  $\mathfrak{g}_{\bar{1}}$ , meaning that  $B(X_i, X_j) = B(Y_i, Y_j) = 0$  and  $B(X_i, Y_j) = \delta_{ij}$ , for all  $1 \leq i, j \leq n$ . Let  $\{p_1, \dots, p_n, q_1, \dots, q_n\}$  be its dual basis. Then the algebra  $\text{Sym}(\mathfrak{g}_{\bar{1}})$  regarded as the polynomial algebra  $\mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n]$  is equipped with the *Poisson bracket*:

$$\{F, G\} = \sum_{i=1}^n \left( \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right), \text{ for all } F, G \in \text{Sym}(\mathfrak{g}_{\bar{1}}).$$

It is well-known that the algebra  $(\text{Sym}(\mathfrak{g}_{\bar{1}}), \{\cdot, \cdot\})$  is a Lie algebra. Now, let  $X \in \mathfrak{g}_{\bar{0}}$  and denote by  $\iota_X$  the derivation of  $\text{Alt}(\mathfrak{g}_{\bar{0}})$  defined by:

$$\iota_X(\Omega)(Z_1, \dots, Z_k) = \Omega(X, Z_1, \dots, Z_k), \forall \Omega \in \text{Alt}^{k+1}(\mathfrak{g}_{\bar{0}}), X, Z_1, \dots, Z_k \in \mathfrak{g}_{\bar{0}} (k \geq 0),$$

and  $\iota_X(1) = 0$ . Let  $\{Z_1, \dots, Z_m\}$  be a fixed orthonormal basis of  $\mathfrak{g}_{\bar{0}}$ . The *super Poisson bracket* on the algebra  $\text{Alt}(\mathfrak{g}_{\bar{0}})$  is defined by (see [PU07] for details):

$$\{\Omega, \Omega'\} = (-1)^{k+1} \sum_{j=1}^m \iota_{Z_j}(\Omega) \wedge \iota_{Z_j}(\Omega'), \forall \Omega \in \text{Alt}^k(\mathfrak{g}_{\bar{0}}), \Omega' \in \text{Alt}(\mathfrak{g}_{\bar{0}}).$$

Remark that the definitions above do not depend on the choice of the basis.

Next, for any  $\Omega \in \text{Alt}^k(\mathfrak{g}_{\bar{0}})$ , we define the map  $\text{ad}_{\mathbb{P}}(\Omega)$  by

$$\text{ad}_{\mathbb{P}}(\Omega)(\Omega') = \{\Omega, \Omega'\}, \forall \Omega' \in \text{Alt}(\mathfrak{g}_{\bar{0}}).$$

It is easy to check that  $\text{ad}_{\mathbb{P}}(\Omega)$  is a super-derivation of degree  $k-2$  of the algebra  $\text{Alt}(\mathfrak{g}_{\bar{0}})$ . One has:

$$\text{ad}_{\mathbb{P}}(\Omega)(\{\Omega', \Omega''\}) = \{\text{ad}_{\mathbb{P}}(\Omega)(\Omega'), \Omega''\} + (-1)^{kk'} \{\Omega', \text{ad}_{\mathbb{P}}(\Omega)(\Omega'')\},$$

for all  $\Omega' \in \text{Alt}^{k'}(\mathfrak{g}_{\bar{0}})$ ,  $\Omega'' \in \text{Alt}(\mathfrak{g}_{\bar{0}})$ . Therefore  $\text{Alt}(\mathfrak{g}_{\bar{0}})$  is a graded Lie algebra for the super-Poisson bracket.

**Definition 1.1.** [MPU09]

The *super  $\mathbb{Z} \times \mathbb{Z}_2$ -Poisson bracket* on  $\mathcal{E}(\mathfrak{g})$  is given by:

$$\{\Omega \otimes F, \Omega' \otimes F'\} = (-1)^{f\omega'} (\{\Omega, \Omega'\} \otimes FF' + (\Omega \wedge \Omega') \otimes \{F, F'\}),$$

for all  $\Omega \in \text{Alt}(\mathfrak{g}_{\bar{0}})$ ,  $\Omega' \in \text{Alt}^{\omega'}(\mathfrak{g}_{\bar{0}})$ ,  $F \in \text{Sym}(\mathfrak{g}_{\bar{1}})^f$ ,  $F' \in \text{Sym}(\mathfrak{g}_{\bar{1}})$ .

By a straightforward computation, it is easy to obtain the following result:

**Proposition 1.2.** *The algebra  $\mathcal{E}(\mathfrak{g})$  is a graded Lie algebra with the super  $\mathbb{Z} \times \mathbb{Z}_2$ -Poisson bracket. More precisely, for all  $A \in \mathcal{E}^{(a,b)}(\mathfrak{g})$ ,  $A' \in \mathcal{E}^{(a',b')}(\mathfrak{g})$  and  $A'' \in \mathcal{E}^{(a'',b'')}(\mathfrak{g})$ :*

- (1)  $\{A', A\} = -(-1)^{aa'+bb'} \{A, A'\}.$
- (2)  $\{\{A, A'\}, A''\} = \{A, \{A', A''\}\} - (-1)^{aa'+bb'} \{A', \{A, A''\}\}.$

Moreover, one has  $\{A, A' \wedge A''\} = \{A, A'\} \wedge A'' + (-1)^{aa'+bb'} A' \wedge \{A, A''\}.$

**1.3. Super-derivations.** Denote by  $\mathcal{L}(\mathcal{E}(\mathfrak{g}))$  the vector space of endomorphisms of  $\mathcal{E}(\mathfrak{g})$ . Let  $\text{ad}_{\mathbb{P}}(A) := \{A, \cdot\}$ , for all  $A \in \mathcal{E}(\mathfrak{g})$ . Then  $\text{ad}_{\mathbb{P}}(A) \in \mathcal{L}(\mathcal{E}(\mathfrak{g}))$  and:

$$\text{ad}_{\mathbb{P}}(\{A, A'\}) = \text{ad}_{\mathbb{P}}(A) \circ \text{ad}_{\mathbb{P}}(A') - (-1)^{aa'+bb'} \text{ad}_{\mathbb{P}}(A') \circ \text{ad}_{\mathbb{P}}(A)$$

for all  $A, A' \in \mathcal{E}^{(a',b')}(\mathfrak{g})$ . The space  $\mathcal{L}(\mathcal{E}(\mathfrak{g}))$  is naturally  $\mathbb{Z} \times \mathbb{Z}_2$ -graded as follows:

$\deg(F) = (n, d)$ ,  $n \in \mathbb{Z}$ ,  $d \in \mathbb{Z}_2$  if  $\deg(F(A)) = (n+a, d+b)$ , where  $A \in \mathcal{E}^{(a,b)}(\mathfrak{g})$ .

We denote by  $\text{End}_f^n(\mathcal{E}(\mathfrak{g}))$  the subspace of endomorphisms of degree  $(n, f)$  of  $\mathcal{L}(\mathcal{E}(\mathfrak{g}))$ . It is clear that if  $A \in \mathcal{E}^{(a,b)}(\mathfrak{g})$  then  $\text{ad}_{\mathbb{P}}(A)$  has degree  $(a-2, b)$ . Moreover, it is known that  $\mathcal{L}(\mathcal{E}(\mathfrak{g}))$  is also a graded Lie algebra, frequently denoted by  $\mathfrak{gl}(\mathcal{E}(\mathfrak{g}))$  and equipped with the Lie super-bracket:

$$[F, G] = F \circ G - (-1)^{np+fg} G \circ F, \quad \forall F \in \text{End}_f^n(\mathcal{E}(\mathfrak{g})), \quad G \in \text{End}_g^p(\mathcal{E}(\mathfrak{g})).$$

Therefore, by Proposition 1.2, we obtain that  $\text{ad}_{\mathbb{P}}$  is a graded Lie algebra homomorphism from  $\mathcal{E}(\mathfrak{g})$  onto  $\mathfrak{gl}(\mathcal{E}(\mathfrak{g}))$ . In other words, one has:

$$\text{ad}_{\mathbb{P}}(\{A, A'\}) = [\text{ad}_{\mathbb{P}}(A), \text{ad}_{\mathbb{P}}(A')], \quad \forall A, A' \in \mathcal{E}(\mathfrak{g}).$$

**Definition 1.3.** A homogeneous endomorphism  $D \in \mathfrak{gl}(\mathcal{E}(\mathfrak{g}))$  of degree  $(n, d)$  is called a *super-derivation* of degree  $(n, d)$  of  $\mathcal{E}(\mathfrak{g})$  (for the super-exterior product) if it satisfies the following condition:

$$D(A \wedge A') = D(A) \wedge A' + (-1)^{na+db} A \wedge D(A'), \quad \forall A \in \mathcal{E}^{(a,b)}(\mathfrak{g}), \quad A' \in \mathcal{E}(\mathfrak{g}).$$



Denote by  $\mathcal{D}_d^n(\mathcal{E}(\mathfrak{g}))$  the space of super-derivations of degree  $(n, d)$  of  $\mathcal{E}(\mathfrak{g})$  then we obtain a  $\mathbb{Z} \times \mathbb{Z}_2$ -gradation of the space of super-derivations  $\mathcal{D}(\mathcal{E}(\mathfrak{g}))$  of  $\mathcal{E}(\mathfrak{g})$ :

$$\mathcal{D}(\mathcal{E}(\mathfrak{g})) = \bigoplus_{(n,d) \in \mathbb{Z} \times \mathbb{Z}_2} \mathcal{D}_d^n(\mathcal{E}(\mathfrak{g}))$$

and  $\mathcal{D}(\mathcal{E}(\mathfrak{g}))$  becomes a graded subalgebra of  $\mathfrak{gl}(\mathcal{E}(\mathfrak{g}))$  [NR66]. Moreover, the last formula in Proposition 1.2 affirms that  $\text{ad}_p(A) \in \mathcal{D}(\mathcal{E}(\mathfrak{g}))$ , for all  $A \in \mathcal{E}(\mathfrak{g})$ .

Another example of a super-derivation in  $\mathcal{D}(\mathcal{E}(\mathfrak{g}))$  is given in [BP89] as follows. Let  $X \in \mathfrak{g}_x$  be a homogeneous element in  $\mathfrak{g}$  of degree  $x$  and define the endomorphism  $\iota_X$  of  $\mathcal{E}(\mathfrak{g})$  by

$$\iota_X(A)(X_1, \dots, X_{a-1}) = (-1)^{xb} A(X, X_1, \dots, X_{a-1})$$

for all  $A \in \mathcal{E}^{(a,b)}(\mathfrak{g})$ ,  $X_1, \dots, X_{a-1} \in V$ . Then one has

$$\iota_X(A \wedge A') = \iota_X(A) \wedge A' + (-1)^{-a+xb} A \wedge \iota_X(A')$$

holds for all  $A \in \mathcal{E}^{(a,b)}(\mathfrak{g})$ ,  $A' \in \mathcal{E}(\mathfrak{g})$ . It means that  $\iota_X$  is a super-derivation of  $\mathcal{E}(\mathfrak{g})$  of degree  $(-1, x)$ . The proof of the following Lemma is straightforward:

**Lemma 1.4.** *Let  $X_{\bar{0}} \in \mathfrak{g}_{\bar{0}}$  and  $X_{\bar{1}} \in \mathfrak{g}_{\bar{1}}$ . Then, for all  $\Omega \otimes F \in \text{Alt}^\omega(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^f(\mathfrak{g}_{\bar{1}})$ :*

- (1)  $\iota_{X_{\bar{0}}}(\Omega \otimes F) = \iota_{X_{\bar{0}}}(\Omega) \otimes F$ ,
- (2)  $\iota_{X_{\bar{1}}}(\Omega \otimes F) = (-1)^\omega \Omega \otimes \iota_{X_{\bar{1}}}(F)$ .

*Remark 1.5.*

- (1) If  $\Omega \in \text{Alt}^\omega(\mathfrak{g}_{\bar{0}})$  then  $\iota_X(\Omega)(X_1, \dots, X_{\omega-1}) = \Omega(X, X_1, \dots, X_{\omega-1})$  for all  $X, X_1, \dots, X_{\omega-1} \in \mathfrak{g}_{\bar{0}}$ . That coincides with the previous definition of  $\iota_X$  on  $\text{Alt}(\mathfrak{g}_{\bar{0}})$ .
- (2) Let  $X$  be an element in a fixed Darboux basis of  $\mathfrak{g}_{\bar{1}}$  and  $p \in \mathfrak{g}_{\bar{1}}^*$  be its dual form. By the Corollary II.1.52 in [Gié04] one has:

$$\iota_X(p^n)(X^{n-1}) = (-1)^n p^n(X^n) = (-1)^n (-1)^{n(n-1)/2} n!.$$

Moreover,  $\frac{\partial p^n}{\partial p}(X^{n-1}) = n(p^{n-1})(X^{n-1}) = (-1)^{(n-1)(n-2)/2} n!$ . It implies that

$$\iota_X(p^n)(X^{n-1}) = -\frac{\partial p^n}{\partial p}(X^{n-1}).$$

Since each  $F \in \text{Sym}^f(\mathfrak{g}_{\bar{1}})$  can be regarded as a polynomial in the variable  $p$ , one has the following property:

$$\iota_X(F) = -\frac{\partial F}{\partial p}, \quad \forall F \in \text{Sym}(\mathfrak{g}_{\bar{1}}).$$

The super-derivations  $\iota_X$  play an important role in the description of the space  $\mathcal{D}(\mathcal{E}(\mathfrak{g}))$  (for details, see [Gié04]). For instance, they can be used to express the super-derivation  $\text{ad}_p(A)$  defined above:

**Proposition 1.6.** *Fix an orthonormal basis  $\{X_0^1, \dots, X_0^m\}$  of  $\mathfrak{g}_0$  and a Darboux basis  $\mathcal{B} = \{X_1^1, \dots, X_1^n, Y_1^1, \dots, Y_1^n\}$  of  $\mathfrak{g}_1$ . Then the super  $\mathbb{Z} \times \mathbb{Z}_2$ -Poisson bracket on  $\mathcal{E}(\mathfrak{g})$  is given by:*

$$\begin{aligned} \{A, A'\} &= (-1)^{\omega+f+1} \sum_{j=1}^m \iota_{X_0^j}(A) \wedge \iota_{X_0^j}(A') \\ &\quad + (-1)^\omega \sum_{k=1}^n \left( \iota_{X_1^k}(A) \wedge \iota_{Y_1^k}(A') - \iota_{Y_1^k}(A) \wedge \iota_{X_1^k}(A') \right) \end{aligned}$$

for all  $A \in \text{Alt}^\omega(\mathfrak{g}_0) \otimes \text{Sym}^f(\mathfrak{g}_1)$  and  $A' \in \mathcal{E}(\mathfrak{g})$ .

*Proof.* Let  $A = \Omega \otimes F \in \text{Alt}^\omega(\mathfrak{g}_0) \otimes \text{Sym}^f(\mathfrak{g}_1)$  and  $A' = \Omega' \otimes F' \in \text{Alt}^{\omega'}(\mathfrak{g}_0) \otimes \text{Sym}^{f'}(\mathfrak{g}_1)$ . The super  $\mathbb{Z} \times \mathbb{Z}_2$ -Poisson bracket of  $A$  and  $A'$  is defined by:

$$\{A, A'\} = (-1)^{f\omega'} (\{\Omega, \Omega'\} \otimes FF' + (\Omega \wedge \Omega') \otimes \{F, F'\}).$$

By the definition of the super Poisson bracket on  $\text{Alt}(\mathfrak{g}_0)$  combined with Lemma 1.4 (1), one has

$$\begin{aligned} \{\Omega, \Omega'\} \otimes FF' &= (-1)^{\omega+1} \sum_{j=1}^m \left( \iota_{X_0^j}(\Omega) \wedge \iota_{X_0^j}(\Omega') \right) \otimes FF' \\ &= (-1)^{f(\omega'-1)+\omega+1} \sum_{j=1}^m \left( \iota_{X_0^j}(\Omega) \otimes F \right) \wedge \left( \iota_{X_0^j}(\Omega') \otimes F' \right) \\ &= (-1)^{f\omega'+\omega+f+1} \sum_{j=1}^m \iota_{X_0^j}(A) \wedge \iota_{X_0^j}(A'). \end{aligned}$$

Let  $\{p_1, \dots, p_n, q_1, \dots, q_n\}$  be the dual basis of  $\mathcal{B}$ . By Remark 1.5 (2), the Poisson bracket on  $\text{Sym}(\mathfrak{g}_1)$  can be expressed by:

$$\{F, F'\} = \sum_{k=1}^n \left( \frac{\partial F}{\partial p_k} \frac{\partial F'}{\partial q_k} - \frac{\partial F}{\partial q_k} \frac{\partial F'}{\partial p_k} \right) = \sum_{k=1}^n \left( \iota_{X_1^k}(F) \iota_{Y_1^k}(F') - \iota_{Y_1^k}(F) \iota_{X_1^k}(F') \right).$$

Combined with Lemma 1.4 (2), we obtain

$$\begin{aligned} (\Omega \wedge \Omega') \otimes \{F, F'\} &= (\Omega \wedge \Omega') \otimes \sum_{k=1}^n \left( \iota_{X_1^k}(F) \iota_{Y_1^k}(F') - \iota_{Y_1^k}(F) \iota_{X_1^k}(F') \right) \\ &= (-1)^{(f-1)\omega'} \sum_{k=1}^n \left( \left( \Omega \otimes \iota_{X_1^k}(F) \right) \wedge \left( \Omega' \otimes \iota_{Y_1^k}(F') \right) - \left( \Omega \otimes \iota_{Y_1^k}(F) \right) \wedge \left( \Omega' \otimes \iota_{X_1^k}(F') \right) \right) \\ &= (-1)^{f\omega'+\omega} \sum_{k=1}^n \left( \iota_{X_1^k}(A) \wedge \iota_{Y_1^k}(A') - \iota_{Y_1^k}(A) \wedge \iota_{X_1^k}(A') \right). \end{aligned}$$

The result follows.  $\square$

Since the bilinear form  $B$  is non-degenerate and even, then there is an (even) isomorphism  $\phi$  from  $\mathfrak{g}$  onto  $\mathfrak{g}^*$  defined by  $\phi(X)(Y) = B(X, Y)$ , for all  $X, Y \in \mathfrak{g}$ .

**Corollary 1.7.** *The following expressions:*

$$(1) \quad \{\alpha, A\} = \iota_{\phi^{-1}(\alpha)}(A),$$

$$(2) \quad \{\alpha, \alpha'\} = B(\phi^{-1}(\alpha), \phi^{-1}(\alpha')),$$

hold for all  $\alpha, \alpha' \in \mathfrak{g}^*$ ,  $A \in \mathcal{E}(\mathfrak{g})$ .

*Proof.*

(1) We apply Proposition 1.6, respectively for  $\alpha = (X_0^i)^* = \phi(X_0^i)$ ,  $i = 1, \dots, m$ ,  $\alpha = (Y_1^l)^* = \phi(X_1^l)$  and  $\alpha = (-X_1^l)^* = \phi(Y_1^l)$ ,  $l = 1, \dots, n$  to obtain the result.

(2) Let  $\alpha \in \mathfrak{g}_x^*$ ,  $\alpha' \in \mathfrak{g}_{x'}^*$  be homogeneous forms in  $\mathfrak{g}^*$ , one has

$$\begin{aligned} \{\alpha, \alpha'\} &= \iota_{\phi^{-1}(\alpha)}(\alpha') = (-1)^{xx'} \alpha'(\phi^{-1}(\alpha)) = (-1)^{xx'} B(\phi^{-1}(\alpha'), \phi^{-1}(\alpha)) \\ &= B(\phi^{-1}(\alpha), \phi^{-1}(\alpha')). \end{aligned}$$

□

In this section, Proposition 1.6 and Corollary 1.7 are enough for our purpose. But as a consequence of Lemma 6.9 in [PU07], one has a more general result of Proposition 1.6 as follows:

**Proposition 1.8.** *Let  $\{X_0^1, \dots, X_0^m\}$  be a basis of  $\mathfrak{g}_0$  and  $\{\alpha_1, \dots, \alpha_m\}$  its dual basis. Let  $\{Y_0^1, \dots, Y_0^m\}$  be the basis of  $\mathfrak{g}_0$  defined by  $Y_0^i = \phi^{-1}(\alpha_i)$ . Set  $\mathcal{B} = \{X_1^1, \dots, X_1^n, Y_1^1, \dots, Y_1^n\}$  be a Darboux basis of  $\mathfrak{g}_1$ . Then the super  $\mathbb{Z} \times \mathbb{Z}_2$ -Poisson bracket on  $\mathcal{E}(\mathfrak{g})$  is given by*

$$\begin{aligned} \{A, A'\} &= (-1)^{\omega+f+1} \sum_{i,j=1}^m B(Y_0^i, Y_0^j) \iota_{X_0^i}(A) \wedge \iota_{X_0^j}(A') \\ &\quad + (-1)^\omega \sum_{k=1}^n \left( \iota_{X_1^k}(A) \wedge \iota_{Y_1^k}(A') - \iota_{Y_1^k}(A) \wedge \iota_{X_1^k}(A') \right) \end{aligned}$$

for all  $A \in \text{Alt}^\omega(\mathfrak{g}_0) \otimes \text{Sym}^f(\mathfrak{g}_1)$  and  $A' \in \mathcal{E}(\mathfrak{g})$ .

**1.4. Super-antisymmetric linear maps.** Consider the vector space

$$\mathcal{E} = \bigoplus_{n \in \mathbb{Z}} \mathcal{E}^n,$$

where  $\mathcal{E}^n = \{0\}$  if  $n \leq -2$ ,  $\mathcal{E}^{-1} = \mathfrak{g}$  and  $\mathcal{E}^n$  is the space of super-antisymmetric  $n+1$ -linear mappings from  $\mathfrak{g}^{n+1}$  onto  $\mathfrak{g}$ . Each of the subspaces  $\mathcal{E}^n$  is  $\mathbb{Z}_2$ -graded then the space  $\mathcal{E}$  is  $\mathbb{Z} \times \mathbb{Z}_2$ -graded by

$$\mathcal{E} = \bigoplus_{f \in \mathbb{Z}_2} \bigoplus_{n \in \mathbb{Z}} \mathcal{E}_f^n.$$

There is a natural isomorphism between the spaces  $\mathcal{E}$  and  $\mathcal{E}(\mathfrak{g}) \otimes \mathfrak{g}$ . Moreover,  $\mathcal{E}$  is a graded Lie algebra, called the *graded Lie algebra* of  $\mathfrak{g}$ . It is isomorphic to  $\mathcal{D}(\mathcal{E}(\mathfrak{g}))$  by the graded Lie algebra isomorphism  $D$  such that if  $F = \Omega \otimes X \in \mathcal{E}_{\omega+x}^n$  then  $D_F = -(-1)^{x\omega} \Omega \wedge \iota_X \in \mathcal{D}_{\omega+x}^n(\mathcal{E}(\mathfrak{g}))$ . For more details on the Lemma below, see for instance, [BP89] and [Gie04].

**Lemma 1.9.**

Fix  $F \in \mathcal{E}_0^1$ , denote by  $d = D_F$  and define the product  $[X, Y] = F(X, Y)$ , for all  $X, Y \in \mathfrak{g}$ . Then one has

- (1)  $d(\phi)(X, Y) = -\phi([X, Y])$ , for all  $X, Y \in \mathfrak{g}$ ,  $\phi \in \mathfrak{g}^*$ .
- (2) The product  $[\cdot, \cdot]$  becomes a Lie super-bracket if and only if  $d^2 = 0$ . In this case,  $d$  is called a super-exterior differential of  $\mathcal{E}(\mathfrak{g})$ .

**1.5. Quadratic Lie superalgebras.** The construction of graded Lie algebras and the super  $\mathbb{Z}_2$ -Poisson bracket above can be applied to the theory of quadratic Lie superalgebras. This later is regarded as a graded version of the quadratic Lie algebra case and we obtain then similar results.

**Definition 1.10.** A quadratic Lie superalgebra  $(\mathfrak{g}, B)$  is a  $\mathbb{Z}_2$ -graded vector space  $\mathfrak{g}$  equipped with a non-degenerate even supersymmetric bilinear form  $B$  and a Lie superalgebra structure such that  $B$  is invariant, i.e.  $B([X, Y], Z) = B(X, [Y, Z])$ , for all  $X, Y, Z \in \mathfrak{g}$ .

**Proposition 1.11.** Let  $(\mathfrak{g}, B)$  be a quadratic Lie superalgebra and define a trilinear form  $I$  on  $\mathfrak{g}$  by

$$I(X, Y, Z) = B([X, Y], Z), \quad \forall X, Y, Z \in \mathfrak{g}.$$

Then one has

- (1)  $I \in \mathcal{E}^{(3, \bar{0})}(\mathfrak{g}) = \text{Alt}^3(\mathfrak{g}_{\bar{0}}) \oplus (\text{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^2(\mathfrak{g}_{\bar{1}}))$ .
- (2)  $d = -\text{ad}_P(I)$ .
- (3)  $\{I, I\} = 0$ .

*Proof.* The assertion (1) follows clearly from the properties of  $B$ . Note that  $B([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}], \mathfrak{g}_{\bar{1}}) = B([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}], \mathfrak{g}_{\bar{1}}) = 0$ .

For (2), fix an orthonormal basis  $\{X_{\bar{0}}^1, \dots, X_{\bar{0}}^m\}$  of  $\mathfrak{g}_{\bar{0}}$  and a Darboux basis  $\{X_{\bar{1}}^1, \dots, X_{\bar{1}}^n, Y_{\bar{1}}^1, \dots, Y_{\bar{1}}^n\}$  of  $\mathfrak{g}_{\bar{1}}$ . Let  $\{\alpha_1, \dots, \alpha_m\}$  and  $\{\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n\}$  be their dual basis, respectively. Then for all  $X, Y \in \mathfrak{g}$ ,  $i = 1, \dots, m$ ,  $l = 1, \dots, n$  we have:

$$\begin{aligned} & \text{ad}_P(I)(\alpha_i)(X, Y) \\ &= \left( \sum_{j=1}^m \iota_{X_{\bar{0}}^j}(I) \wedge \iota_{X_{\bar{0}}^j}(\alpha_i) - \sum_{k=1}^n \left( \iota_{X_{\bar{1}}^k}(I) \wedge \iota_{Y_{\bar{1}}^k}(\alpha_i) - \iota_{Y_{\bar{1}}^k}(I) \wedge \iota_{X_{\bar{1}}^k}(\alpha_i) \right) \right) (X, Y) \\ &= \left( \sum_{j=1}^m \iota_{X_{\bar{0}}^j}(I) \wedge \iota_{X_{\bar{0}}^j}(\alpha_i) \right) (X, Y) = \left( \iota_{X_{\bar{0}}^i}(I) \wedge \iota_{X_{\bar{0}}^i}(\alpha_i) \right) (X, Y) \\ &= B(X_{\bar{0}}^i, [X, Y]) = \alpha_i([X, Y]) = -d(\alpha_i)(X, Y), \end{aligned}$$

$$\begin{aligned}
& \text{ad}_P(I)(\beta_l)(X, Y) \\
&= \left( \sum_{j=1}^m \iota_{X_0^j}(I) \wedge \iota_{X_0^j}(\beta_l) - \sum_{k=1}^n \left( \iota_{X_1^k}(I) \wedge \iota_{Y_1^k}(\beta_l) - \iota_{Y_1^k}(I) \wedge \iota_{X_1^k}(\beta_l) \right) \right) (X, Y) \\
&= \left( \sum_{k=1}^n \iota_{Y_1^k}(I) \wedge \iota_{X_1^k}(\beta_l) \right) (X, Y) = \left( \iota_{Y_1^l}(I) \wedge \iota_{X_1^l}(\beta_l) \right) (X, Y) \\
&= -\iota_{Y_1^l}(I)(X, Y) = -B(Y_1^l, [X, Y]) = \beta_l([X, Y]) = -d(\beta_l)(X, Y).
\end{aligned}$$

Similarly,  $\text{ad}_P(I)(\gamma_l) = -d(\gamma_l)$  for  $1 \leq l \leq n$ . Therefore,  $d = -\text{ad}_P(I)$ .

Moreover,  $\text{ad}_P(\{I, I\}) = [\text{ad}_P(I), \text{ad}_P(I)] = [d, d] = 2d^2 = 0$ . Therefore, for all  $1 \leq i \leq m$ ,  $1 \leq j, k \leq n$  one has  $\{\alpha_i, \{I, I\}\} = \{\beta_j, \{I, I\}\} = \{\gamma_k, \{I, I\}\} = 0$ . Those imply  $\iota_X(\{I, I\}) = 0$  for all  $X \in \mathfrak{g}$  and hence, we obtain  $\{I, I\} = 0$ .  $\square$

Conversely, let  $\mathfrak{g}$  be a quadratic  $\mathbb{Z}_2$ -graded vector space equipped with a bilinear form  $B$  and  $I$  be an element in  $\mathcal{E}^{(3, \bar{0})}(\mathfrak{g})$ . Define  $d = -\text{ad}_P(I)$  then  $d \in \mathcal{D}_0^1(\mathcal{E}(\mathfrak{g}))$ . Therefore,  $d^2 = 0$  if and only if  $\{I, I\} = 0$ . Let  $F$  be the structure in  $\mathfrak{g}$  corresponding to  $d$  by the isomorphism  $D$  in Lemma 1.9, one has

**Proposition 1.12.**  *$F$  becomes a Lie superalgebra structure if and only if  $\{I, I\} = 0$ . In this case, with the notation  $[X, Y] := F(X, Y)$  one has:*

$$I(X, Y, Z) = B([X, Y], Z), \quad \forall X, Y, Z \in \mathfrak{g}.$$

Moreover, the bilinear form  $B$  is invariant.

*Proof.* We need to prove that if  $F$  is a Lie superalgebra structure then  $I(X, Y, Z) = B([X, Y], Z)$ , for all  $X, Y, Z \in \mathfrak{g}$ . Indeed, let  $\{X_0^1, \dots, X_0^m\}$  be an orthonormal basis of  $\mathfrak{g}_0$  and  $\{X_1^1, \dots, X_1^n, Y_1^1, \dots, Y_1^n\}$  be a Darboux basis of  $\mathfrak{g}_1$  then one has

$$d = -\text{ad}_P(I) = -\sum_{j=1}^m \iota_{X_0^j}(I) \wedge \iota_{X_0^j} + \sum_{k=1}^n \iota_{X_1^k}(I) \wedge \iota_{Y_1^k} - \sum_{k=1}^n \iota_{Y_1^k}(I) \wedge \iota_{X_1^k}.$$

It implies that

$$F = \sum_{j=1}^m \iota_{X_0^j}(I) \otimes X_0^j + \sum_{k=1}^n \iota_{X_1^k}(I) \otimes Y_1^k - \sum_{k=1}^n \iota_{Y_1^k}(I) \otimes X_1^k.$$

Therefore, for all  $i$  we obtain

$$B([X, Y], X_0^i) = \iota_{X_0^i}(I)(X, Y) = I(X_0^i, X, Y) = I(X, Y, X_0^i),$$

$$B([X, Y], X_1^i) = -\iota_{X_1^i}(I)(X, Y) = -I(X_1^i, X, Y) = I(X, Y, X_1^i),$$

$$\text{and } B([X, Y], Y_1^i) = -\iota_{Y_1^i}(I)(X, Y) = -I(Y_1^i, X, Y) = I(X, Y, Y_1^i).$$

These show that  $I(X, Y, Z) = B([X, Y], Z)$ , for all  $X, Y, Z \in \mathfrak{g}$ . Since  $I$  is super-antisymmetric and  $B$  is supersymmetric, then one can show that  $B$  is invariant.  $\square$

The two previous propositions show that on a quadratic  $\mathbb{Z}_2$ -graded vector space  $(\mathfrak{g}, B)$ , quadratic Lie superalgebra structures with the same  $B$  are in one to one correspondence with elements  $I \in \mathcal{E}^{(3, \overline{0})}(\mathfrak{g})$  satisfying  $\{I, I\} = 0$  and such that the super-exterior differential of  $\mathcal{E}(\mathfrak{g})$  is  $d = -\text{ad}_{\mathfrak{p}}(I)$ . This correspondence provides an approach to the theory of quadratic Lie superalgebras through  $I$ .

**Definition 1.13.** Given a quadratic Lie superalgebra  $(\mathfrak{g}, B)$ . The element  $I$  defined as above is also an invariant of  $\mathfrak{g}$  since  $\mathcal{L}_X(I) = 0$ , for all  $X \in \mathfrak{g}$  where  $\mathcal{L}_X = D(\text{ad}_{\mathfrak{g}}(X))$  is the Lie super-derivation of  $\mathfrak{g}$ . Therefore,  $I$  is called the *associated invariant* of  $\mathfrak{g}$ .

The following Lemma is a simple, yet interesting result.

**Lemma 1.14.** Let  $(\mathfrak{g}, B)$  be a quadratic Lie superalgebra and  $I$  be its associated invariant. Then  $\iota_X(I) = 0$  if and only if  $X \in \mathcal{Z}(\mathfrak{g})$ .

*Proof.* Since  $\iota_X(I)(\mathfrak{g}, \mathfrak{g}) = B(X, [\mathfrak{g}, \mathfrak{g}])$  and  $\mathcal{Z}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]^\perp$  where  $[\mathfrak{g}, \mathfrak{g}]^\perp$  denotes the orthogonal subspace of  $[\mathfrak{g}, \mathfrak{g}]$ . We have then  $\iota_X(I) = 0$  if and only if  $X \in \mathcal{Z}(\mathfrak{g})$ .  $\square$

**Definition 1.15.** Let  $(\mathfrak{g}, B)$  and  $(\mathfrak{g}', B')$  be two quadratic Lie superalgebras. We say that  $(\mathfrak{g}, B)$  and  $(\mathfrak{g}', B')$  are *isometrically isomorphic* (or *i-isomorphic*) if there exists a Lie superalgebra isomorphism  $A$  from  $\mathfrak{g}$  onto  $\mathfrak{g}'$  satisfying

$$B'(A(X), A(Y)) = B(X, Y), \quad \forall X, Y \in \mathfrak{g}.$$

In other words,  $A$  is an i-isomorphism if it is a (necessarily even) Lie superalgebra isomorphism and an isometry. We write  $\mathfrak{g} \stackrel{i}{\simeq} \mathfrak{g}'$ .

Note that two isomorphic quadratic Lie superalgebras  $(\mathfrak{g}, B)$  and  $(\mathfrak{g}', B')$  are not necessarily i-isomorphic by the example below:

**Example 1.16.** Let  $\mathfrak{g} = \mathfrak{osp}(1, 2)$  and  $B$  its Killing form. Recall that  $\mathfrak{g}_{\overline{0}} = \mathfrak{o}(3)$ . Consider another bilinear form  $B' = \lambda B$ ,  $\lambda \in \mathbb{C}$ ,  $\lambda \neq 0$ . In this case,  $(\mathfrak{g}, B)$  and  $(\mathfrak{g}, \lambda B)$  can not be i-isomorphic if  $\lambda \neq 1$  since  $(\mathfrak{g}_{\overline{0}}, B)$  and  $(\mathfrak{g}_{\overline{0}}, \lambda B)$  are not i-isomorphic.

## 2. THE DUP-NUMBER OF QUADRATIC LIE SUPERALGEBRAS

Let  $(\mathfrak{g}, B)$  be a quadratic Lie superalgebra and  $I$  be its associated invariant. Then by Proposition 1.11 we have a decomposition

$$I = I_0 + I_1$$

where  $I_0 \in \text{Alt}^3(\mathfrak{g}_{\overline{0}})$  and  $I_1 \in \text{Alt}^1(\mathfrak{g}_{\overline{0}}) \otimes \text{Sym}^2(\mathfrak{g}_{\overline{1}})$ . Since  $\{I, I\} = 0$ , then  $\{I_0, I_0\} = 0$ . It means that  $\mathfrak{g}_{\overline{0}}$  is a quadratic Lie algebra with the associated 3-form  $I_0$ , a rather obvious result. It is easy to see that  $\mathfrak{g}_{\overline{0}}$  is Abelian (resp.  $[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] = \{0\}$ ) if and only if  $I_0 = 0$  (resp.  $I_1 = 0$ ). These cases will be fully studied in the sequel. Define the following subspaces of  $\mathfrak{g}^*$ :

$$\begin{aligned} \mathcal{V}_I &= \{\alpha \in \mathfrak{g}^* \mid \alpha \wedge I = 0\}, \\ \mathcal{V}_{I_0} &= \{\alpha \in \mathfrak{g}_{\overline{0}}^* \mid \alpha \wedge I_0 = 0\}, \\ \mathcal{V}_{I_1} &= \{\alpha \in \mathfrak{g}_{\overline{0}}^* \mid \alpha \wedge I_1 = 0\}. \end{aligned}$$

**Lemma 2.1.** *Let  $\mathfrak{g}$  be a non-Abelian quadratic Lie superalgebra then one has*

- (1)  $\dim(\mathcal{V}_I) \in \{0, 1, 3\}$ ,
- (2)  $\dim(\mathcal{V}_I) = 3$  if and only if  $I_1 = 0$ ,  $\mathfrak{g}_{\bar{0}}$  is non-Abelian and  $I_0$  is decomposable in  $\text{Alt}^3(\mathfrak{g}_{\bar{0}})$ .

*Proof.* Let  $\alpha = \alpha_0 + \alpha_1 \in \mathfrak{g}_{\bar{0}}^* \oplus \mathfrak{g}_{\bar{1}}^*$  then one has

$$\alpha \wedge I = \alpha_0 \wedge I_0 + \alpha_0 \wedge I_1 + \alpha_1 \wedge I_0 + \alpha_1 \wedge I_1,$$

where  $\alpha_0 \wedge I_0 \in \text{Alt}^4(\mathfrak{g}_{\bar{0}})$ ,  $\alpha_0 \wedge I_1 \in \text{Alt}^2(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^2(\mathfrak{g}_{\bar{1}})$ ,  $\alpha_1 \wedge I_0 \in \text{Alt}^3(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^1(\mathfrak{g}_{\bar{1}})$  and  $\alpha_1 \wedge I_1 \in \text{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^3(\mathfrak{g}_{\bar{1}})$ .

Hence,  $\alpha \wedge I = 0$  if and only if  $\alpha_1 = 0$  and  $\alpha_0 \wedge I_0 = \alpha_0 \wedge I_1 = 0$ . It means that  $\mathcal{V}_I = \mathcal{V}_{I_0} \cap \mathcal{V}_{I_1}$ . If  $I_0 \neq 0$  then  $\dim(\mathcal{V}_{I_0}) \in \{0, 1, 3\}$  and if  $I_1 \neq 0$  then  $\dim(\mathcal{V}_{I_1}) \in \{0, 1\}$ . Therefore,  $\dim(\mathcal{V}_I) \in \{0, 1, 3\}$  and  $\dim(\mathcal{V}_I) = 3$  if and only if  $I_1 = 0$  and  $\dim(\mathcal{V}_{I_0}) = 3$ .  $\square$

The previous Lemma allows us to introduce the notion of *dup-number* for quadratic Lie superalgebras as we did for quadratic Lie algebras.

**Definition 2.2.** Let  $(\mathfrak{g}, B)$  be a non-Abelian quadratic Lie superalgebra and  $I$  be its associated invariant. The *dup-number*  $\text{dup}(\mathfrak{g})$  is defined by

$$\text{dup}(\mathfrak{g}) = \dim(\mathcal{V}_I).$$

Given a subspace  $W$  of  $\mathfrak{g}$ , if  $W$  is *non-degenerate* (with respect to the bilinear form  $B$ ), that is, if the restriction of  $B$  on  $W \times W$  is non-degenerate, then the orthogonal subspace  $W^\perp$  of  $W$  is also non-degenerate. In this case, we use the notation

$$\mathfrak{g} = W \overset{\perp}{\oplus} W^\perp.$$

The decomposition result below is a generalization of the quadratic Lie algebra case. Its proof can be found in [PU07] and [DPU].

**Proposition 2.3.** *Let  $(\mathfrak{g}, B)$  be a non-Abelian quadratic Lie superalgebra. Then there are a central ideal  $\mathfrak{z}$  and an ideal  $\mathfrak{l} \neq \{0\}$  such that:*

- (1)  $\mathfrak{g} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$  where  $(\mathfrak{z}, B|_{\mathfrak{z} \times \mathfrak{z}})$  and  $(\mathfrak{l}, B|_{\mathfrak{l} \times \mathfrak{l}})$  are quadratic Lie superalgebras. Moreover,  $\mathfrak{l}$  is non-Abelian.
- (2) The center  $\mathcal{Z}(\mathfrak{l})$  is totally isotropic, i.e.  $\mathcal{Z}(\mathfrak{l}) \subset [\mathfrak{l}, \mathfrak{l}]$ .
- (3) Let  $\mathfrak{g}'$  be a quadratic Lie superalgebra and  $A : \mathfrak{g} \rightarrow \mathfrak{g}'$  be a Lie superalgebra isomorphism. Then

$$\mathfrak{g}' = \mathfrak{z}' \overset{\perp}{\oplus} \mathfrak{l}'$$

where  $\mathfrak{z}' = A(\mathfrak{z})$  is central,  $\mathfrak{l}' = A(\mathfrak{l})^\perp$ ,  $\mathcal{Z}(\mathfrak{l}')$  is totally isotropic and  $\mathfrak{l}$  and  $\mathfrak{l}'$  are isomorphic. Moreover if  $A$  is an *i-isomorphism*, then  $\mathfrak{l}$  and  $\mathfrak{l}'$  are *i-isomorphic*.

The Lemma below shows that the previous decomposition has a good behavior with respect to the dup-number.

**Lemma 2.4.** *Let  $\mathfrak{g}$  be a non-Abelian quadratic Lie superalgebra. Write  $\mathfrak{g} = \mathfrak{z} \overset{\perp}{\oplus} \mathfrak{l}$  as in Proposition 2.3 then  $\text{dup}(\mathfrak{g}) = \text{dup}(\mathfrak{l})$ .*



*Proof.* Since  $[\mathfrak{z}, \mathfrak{g}] = \{0\}$  then  $I \in \mathcal{E}^{(3, \bar{0})}(\mathfrak{l})$ . Let  $\alpha \in \mathfrak{g}^*$  such that  $\alpha \wedge I = 0$ , we show that  $\alpha \in \mathfrak{l}^*$ . Assume that  $\alpha = \alpha_1 + \alpha_2$ , where  $\alpha_1 \in \mathfrak{z}^*$  and  $\alpha_2 \in \mathfrak{l}^*$ . Since  $\alpha \wedge I = 0$ ,  $\alpha_1 \wedge I \in \mathcal{E}(\mathfrak{z}) \otimes \mathcal{E}(\mathfrak{l})$  and  $\alpha_2 \wedge I \in \mathcal{E}(\mathfrak{l})$  then one has  $\alpha_1 \wedge I = 0$ . Therefore,  $\alpha_1 = 0$  since  $I$  is nonzero in  $\mathcal{E}^{(3, \bar{0})}(\mathfrak{l})$ . That means  $\alpha \in \mathfrak{l}^*$  and then  $\text{dup}(\mathfrak{g}) = \text{dup}(\mathfrak{l})$ .  $\square$

Clearly,  $\mathfrak{z} = \{0\}$  if and only if  $\mathcal{Z}(\mathfrak{g})$  is totally isotropic. By the above Lemma, it is enough to restrict our study on the dup-number of non-Abelian quadratic Lie superalgebras with totally isotropic center.

**Definition 2.5.** A quadratic Lie superalgebra  $\mathfrak{g}$  is *reduced* if it satisfies:

- (1)  $\mathfrak{g} \neq \{0\}$
- (2)  $\mathcal{Z}(\mathfrak{g})$  is totally isotropic.

Notice that a reduced quadratic Lie superalgebra is necessarily non-Abelian.

**Definition 2.6.** Let  $\mathfrak{g}$  be a non-Abelian quadratic Lie superalgebra. We say that:

- (1)  $\mathfrak{g}$  is an *ordinary* quadratic Lie superalgebra if  $\text{dup}(\mathfrak{g}) = 0$ .
- (2)  $\mathfrak{g}$  is a *singular* quadratic Lie superalgebra if  $\text{dup}(\mathfrak{g}) \geq 1$ .
  - (i)  $\mathfrak{g}$  is a *singular* quadratic Lie superalgebra of *type*  $S_1$  if  $\text{dup}(\mathfrak{g}) = 1$ .
  - (ii)  $\mathfrak{g}$  is a *singular* quadratic Lie superalgebra of *type*  $S_3$  if  $\text{dup}(\mathfrak{g}) = 3$ .

By Lemma 2.1, if  $\mathfrak{g}$  is a singular quadratic Lie superalgebra of type  $S_3$  then  $I = I_0$  is decomposable in  $\text{Alt}^3(\mathfrak{g}_{\bar{0}})$ . One has  $I(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}) = B([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}], \mathfrak{g}_{\bar{1}}) = 0$ . It implies  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}] = \{0\}$  since  $B$  is non-degenerate. Hence in this case,  $\mathfrak{g}_{\bar{1}}$  is a central ideal,  $\mathfrak{g}_{\bar{0}}$  is a singular quadratic Lie algebra of type  $S_3$  and then the classification is known in [PU07]. Therefore, we are mainly interested in singular quadratic Lie superalgebras of type  $S_1$ .

Before proceeding, we give other simple properties of singular quadratic Lie superalgebras:

**Proposition 2.7.** Let  $(\mathfrak{g}, B)$  be a singular quadratic Lie superalgebra. If  $\mathfrak{g}_{\bar{0}}$  is non-Abelian then  $\mathfrak{g}_{\bar{0}}$  is a singular quadratic Lie algebra.

*Proof.* By the proof of Lemma 2.1, one has  $\mathcal{V}_I = \mathcal{V}_{I_0} \cap \mathcal{V}_{I_1}$ . Therefore,  $\dim(\mathcal{V}_{I_0}) \geq 1$ . It means that  $\mathfrak{g}_{\bar{0}}$  is a singular quadratic Lie algebra.  $\square$

Given  $(\mathfrak{g}, B)$  a singular quadratic Lie superalgebra of type  $S_1$ . Fix  $\alpha \in \mathcal{V}_I$  and choose  $\Omega_0 \in \text{Alt}^2(\mathfrak{g}_{\bar{0}})$ ,  $\Omega_1 \in \text{Sym}^2(\mathfrak{g}_{\bar{1}})$  such that  $I = \alpha \wedge \Omega_0 + \alpha \otimes \Omega_1$ . Then one has

$$\{I, I\} = \{\alpha \wedge \Omega_0, \alpha \wedge \Omega_0\} + 2\{\alpha \wedge \Omega_0, \alpha\} \otimes \Omega_1 + \{\alpha, \alpha\} \otimes \Omega_1 \Omega_1.$$

By the equality  $\{I, I\} = 0$ , one has  $\{\alpha \wedge \Omega_0, \alpha \wedge \Omega_0\} = 0$ ,  $\{\alpha, \alpha\} = 0$  and  $\{\alpha, \alpha \wedge \Omega_0\} = 0$ . These imply that  $\{\alpha, I\} = 0$ . Hence, if we set  $X_0 = \phi^{-1}(\alpha)$  then  $X_0 \in \mathcal{Z}(\mathfrak{g})$  and  $B(X_0, X_0) = 0$  (Corollary 1.7 and Lemma 1.14).

**Proposition 2.8.** Let  $(\mathfrak{g}, B)$  be a singular quadratic Lie superalgebra. If  $\mathfrak{g}$  is reduced then  $\mathfrak{g}_{\bar{0}}$  is reduced.

*Proof.* As above, if  $\mathfrak{g}$  is a singular quadratic Lie superalgebra of type  $S_3$  then  $\mathfrak{g}_\tau$  is central. By  $\mathfrak{g}$  reduced and  $\mathfrak{g}_\tau$  non-degenerate,  $\mathfrak{g}_\tau$  must be zero and then the result follows.

If  $\mathfrak{g}$  is a singular quadratic Lie superalgebra of type  $S_1$ . Assume that  $\mathfrak{g}_\tau$  is not reduced, i.e.  $\mathfrak{g}_\tau = \mathfrak{z} \oplus \mathfrak{l}$  where  $\mathfrak{z}$  is a non-trivial central ideal of  $\mathfrak{g}_\tau$ , there is  $X \in \mathfrak{z}$  such that  $B(X, X) = 1$ . Since  $\mathfrak{g}$  is singular of type  $S_1$  then  $\mathfrak{g}_\tau$  is also singular. Hence, the element  $X_0$  defined as above must be in  $\mathfrak{l}$  and  $I_0 = \alpha \wedge \Omega_0 \in \text{Alt}^3(\mathfrak{l})$  (see [DPU] for details). We also have  $B(X, X_0) = 0$ .

Let  $\beta = \phi(X)$  so  $\iota_X(I) = \{\beta, I\} = \{\beta, \alpha \wedge \Omega_0 + \alpha \otimes \Omega_1\} = 0$ . That means  $X \in \mathcal{Z}(\mathfrak{g})$ . This is a contradiction since  $\mathfrak{g}$  is reduced. Hence  $\mathfrak{g}_\tau$  must be reduced.  $\square$

**Lemma 2.9.** *Let  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  be non-Abelian quadratic Lie superalgebras. Then  $\mathfrak{g}_1 \oplus \mathfrak{g}_2$  is an ordinary quadratic Lie algebra.*

*Proof.* Set  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ . Denote by  $I, I_1$  and  $I_2$  their non-trivial associated invariants, respectively. One has  $\mathcal{E}(\mathfrak{g}) = \mathcal{E}(\mathfrak{g}_1) \otimes \mathcal{E}(\mathfrak{g}_2)$ ,  $\mathcal{E}^k(\mathfrak{g}) = \bigoplus_{r+s=k} \mathcal{E}^r(\mathfrak{g}_1) \otimes \mathcal{E}^s(\mathfrak{g}_2)$  and  $I = I_1 + I_2$  where  $I_1 \in \mathcal{E}^3(\mathfrak{g}_1), I_2 \in \mathcal{E}^3(\mathfrak{g}_2)$ . Therefore, if  $\alpha = \alpha_1 + \alpha_2 \in \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$  such that  $\alpha \wedge I = 0$  then  $\alpha_1 = \alpha_2 = 0$ .  $\square$

**Definition 2.10.** A quadratic Lie superalgebra  $\mathfrak{g}$  is *indecomposable* if  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , with  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  ideals of  $\mathfrak{g}$ , then  $\mathfrak{g}_1$  or  $\mathfrak{g}_2 = \{0\}$ .

The following result shows that indecomposable and reduced notions are equivalent for singular quadratic Lie superalgebras.

**Proposition 2.11.** *Let  $\mathfrak{g}$  be a singular quadratic Lie superalgebra. Then  $\mathfrak{g}$  is reduced if and only if  $\mathfrak{g}$  is indecomposable.*

*Proof.* If  $\mathfrak{g}$  is indecomposable then it is obvious that  $\mathfrak{g}$  is reduced. If  $\mathfrak{g}$  is reduced, assume that  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , with  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  ideals of  $\mathfrak{g}$ , then  $\mathcal{Z}(\mathfrak{g}_i) \subset [\mathfrak{g}_i, \mathfrak{g}_i]$  for  $i = 1, 2$ . Therefore,  $\mathfrak{g}_i$  is reduced or  $\mathfrak{g}_i = \{0\}$ . If  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are both reduced, by Lemma 2.9, then  $\mathfrak{g}$  is ordinary. Hence  $\mathfrak{g}_1$  or  $\mathfrak{g}_2 = \{0\}$ .  $\square$

### 3. ELEMENTARY QUADRATIC LIE SUPERALGEBRAS

In this section, we consider the first non-trivial case of singular quadratic Lie superalgebras: elementary quadratic Lie superalgebras. We begin with the following definition.

**Definition 3.1.** Let  $\mathfrak{g}$  be a quadratic Lie superalgebra and  $I$  be its associated invariant. We say that  $\mathfrak{g}$  is an *elementary* quadratic Lie superalgebra if  $I$  is decomposable.

Keep notations as in Section 2. If  $I = I_0 + I_1$  is decomposable, where  $I_0 \in \text{Alt}^3(\mathfrak{g}_\tau)$  and  $I_1 \in \text{Alt}^1(\mathfrak{g}_\tau) \otimes \text{Sym}^2(\mathfrak{g}_\tau)$  then it is obvious that  $I_0$  or  $I_1$  is zero. The case  $I_1 = 0$ , i.e.  $I$  decomposable in  $\text{Alt}^3(\mathfrak{g}_\tau)$ , corresponds to singular quadratic Lie superalgebras of type  $S_3$  and then there is nothing to do. Now we assume  $I$  is a nonzero decomposable element in  $\text{Alt}^1(\mathfrak{g}_\tau) \otimes \text{Sym}^2(\mathfrak{g}_\tau)$  then  $I$  can be written by:

$$I = \alpha \otimes pq$$

where  $\alpha \in \mathfrak{g}_0^*$  and  $p, q \in \mathfrak{g}_1^*$ . It is clear that  $\mathfrak{g}$  is a singular quadratic Lie superalgebra of type  $S_1$ .

**Lemma 3.2.** *Let  $\mathfrak{g}$  be a reduced elementary quadratic Lie superalgebra having  $I = \alpha \otimes pq$  where  $\alpha \in \mathfrak{g}_0^*$  and  $p, q \in \mathfrak{g}_1^*$ . Set  $X_{\bar{0}} = \phi^{-1}(\alpha)$  then one has:*

- (1)  $\dim(\mathfrak{g}_{\bar{0}}) = 2$  and  $\mathfrak{g}_{\bar{0}} \cap \mathcal{Z}(\mathfrak{g}) = \mathbb{C}X_{\bar{0}}$ .
- (2) Let  $X_{\bar{1}} = \phi^{-1}(p)$ ,  $Y_{\bar{1}} = \phi^{-1}(q)$  and  $U = \text{span}\{X_{\bar{1}}, Y_{\bar{1}}\}$  then
  - (i)  $\dim(\mathfrak{g}_{\bar{1}}) = 2$  if  $\dim(U) = 1$  or  $U$  is non-degenerate.
  - (ii)  $\dim(\mathfrak{g}_{\bar{1}}) = 4$  if  $U$  is totally isotropic.

*Proof.*

- (1) Let  $\beta$  be an element in  $\mathfrak{g}_0^*$ . It is easy to see that  $\{\beta, \alpha\} = 0$  if and only if  $\{\beta, I\} = 0$ , equivalently  $\phi^{-1}(\beta) \in \mathcal{Z}(\mathfrak{g})$ . Therefore,  $(\phi^{-1}(\alpha))^\perp \cap \mathfrak{g}_{\bar{0}} \subset \mathcal{Z}(\mathfrak{g})$ . It means that  $\dim(\mathfrak{g}_{\bar{0}}) \leq 2$  since  $\mathfrak{g}$  is reduced (see [Bou59]). Moreover,  $X_{\bar{0}} = \phi^{-1}(\alpha)$  is isotropic then  $\dim(\mathfrak{g}_{\bar{0}}) = 2$ . If  $\dim(\mathfrak{g}_{\bar{0}} \cap \mathcal{Z}(\mathfrak{g})) = 2$  then  $\mathfrak{g}_{\bar{0}} \subset \mathcal{Z}(\mathfrak{g})$ . Since  $B$  is invariant we obtain  $\mathfrak{g}$  Abelian (a contradiction). Therefore,  $\mathfrak{g}_{\bar{0}} \cap \mathcal{Z}(\mathfrak{g}) = \mathbb{C}X_{\bar{0}}$ .
- (2) It is obvious that  $\dim(\mathfrak{g}_{\bar{1}}) \geq 2$ . If  $\dim(U) = 1$  then  $U$  is a totally isotropic subspace of  $\mathfrak{g}_{\bar{1}}$  so there exists a one-dimensional subspace  $V$  of  $\mathfrak{g}_{\bar{1}}$  such that  $B$  is non-degenerate on  $U \oplus V$  (see [Bou59]). Let  $\mathfrak{g}_{\bar{1}} = (U \oplus V) \oplus W$  where  $W = (U \oplus V)^\perp$  then for all  $f \in \phi(W)$  one has:

$$\{f, I\} = \{f, \alpha \otimes pq\} = -\alpha \otimes (\{f, p\}q + p\{f, q\}) = 0.$$

Therefore,  $W \subset \mathcal{Z}(\mathfrak{g})$ . Since  $B$  is non-degenerate on  $W$  and  $\mathfrak{g}$  is reduced then  $W = \{0\}$ .

If  $\dim(U) = 2$  then  $U$  is non-degenerate or totally isotropic. If  $U$  is non-degenerate, let  $\mathfrak{g}_{\bar{1}} = U \oplus W$  where  $W = U^\perp$ . If  $U$  is totally isotropic, let  $\mathfrak{g}_{\bar{1}} = (U \oplus V) \oplus W$  where  $W = (U \oplus V)^\perp$  in  $\mathfrak{g}_{\bar{1}}$  and  $B$  is non-degenerate on  $U \oplus V$ . In the both cases, similarly as above, one has  $W$  a non-degenerate central ideal so  $W = \{0\}$ . Therefore,  $\dim(\mathfrak{g}_{\bar{1}}) = \dim(U) = 2$  if  $U$  is non-degenerate and  $\dim(\mathfrak{g}_{\bar{1}}) = \dim(U \oplus V) = 4$  if  $U$  is totally isotropic. □

In the sequel, we obtain the classification result.

**Proposition 3.3.** *Let  $\mathfrak{g}$  be a reduced elementary quadratic Lie superalgebra then  $\mathfrak{g}$  is isomorphic to one of the following Lie superalgebras:*

- (1)  $\mathfrak{g}_i$  ( $3 \leq i \leq 6$ ) the reduced singular quadratic Lie algebras of type  $S_3$  given in [PU07].
- (2)  $\mathfrak{g}_{4,1}^s = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \oplus (\mathbb{C}X_{\bar{1}} \oplus \mathbb{C}Z_{\bar{1}})$  where  $\mathfrak{g}_{\bar{0}} = \text{span}\{X_{\bar{0}}, Y_{\bar{0}}\}$ ,  $\mathfrak{g}_{\bar{1}} = \text{span}\{X_{\bar{1}}, Z_{\bar{1}}\}$ , the bilinear form  $B$  is defined by

$$B(X_{\bar{0}}, Y_{\bar{0}}) = B(X_{\bar{1}}, Z_{\bar{1}}) = 1,$$

the other are zero and the Lie super-bracket is given by

$$[Z_{\bar{1}}, Z_{\bar{1}}] = -2X_{\bar{0}}, \quad [Y_{\bar{0}}, Z_{\bar{1}}] = -2X_{\bar{1}},$$

the other are trivial.

- (3)  $\mathfrak{g}_{4,2}^s = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \oplus (\mathbb{C}X_{\bar{1}} \oplus \mathbb{C}Y_{\bar{1}})$  where  $\mathfrak{g}_{\bar{0}} = \text{span}\{X_{\bar{0}}, Y_{\bar{0}}\}$ ,  $\mathfrak{g}_{\bar{1}} = \text{span}\{X_{\bar{1}}, Y_{\bar{1}}\}$ , the bilinear form  $B$  is defined by

$$B(X_{\bar{0}}, Y_{\bar{0}}) = B(X_{\bar{1}}, Y_{\bar{1}}) = 1,$$

the other are zero and the Lie super-bracket is given by

$$[X_{\bar{1}}, Y_{\bar{1}}] = X_{\bar{0}}, \quad [Y_{\bar{0}}, X_{\bar{1}}] = X_{\bar{1}}, \quad [Y_{\bar{0}}, Y_{\bar{1}}] = -Y_{\bar{1}},$$

the other are trivial.

- (4)  $\mathfrak{g}_6^s = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \oplus (\mathbb{C}X_{\bar{1}} \oplus \mathbb{C}Y_{\bar{1}} \oplus \mathbb{C}Z_{\bar{1}} \oplus \mathbb{C}T_{\bar{1}})$  where  $\mathfrak{g}_{\bar{0}} = \text{span}\{X_{\bar{0}}, Y_{\bar{0}}\}$ ,  $\mathfrak{g}_{\bar{1}} = \text{span}\{X_{\bar{1}}, Y_{\bar{1}}, Z_{\bar{1}}, T_{\bar{1}}\}$ , the bilinear form  $B$  is defined by

$$B(X_{\bar{0}}, Y_{\bar{0}}) = B(X_{\bar{1}}, Z_{\bar{1}}) = B(Y_{\bar{1}}, T_{\bar{1}}) = 1,$$

the other are zero and the Lie super-bracket is given by

$$[Z_{\bar{1}}, T_{\bar{1}}] = -X_{\bar{0}}, \quad [Y_{\bar{0}}, Z_{\bar{1}}] = -Y_{\bar{1}}, \quad [Y_{\bar{0}}, T_{\bar{1}}] = -X_{\bar{1}},$$

the other are trivial.

*Proof.*

- (1) This statement corresponds to the case where  $I$  is a decomposable 3-form in  $\text{Alt}^3(\mathfrak{g}_{\bar{0}})$ . Therefore, the result is obvious.

Assume that  $I = \alpha \otimes pq \in \text{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^2(\mathfrak{g}_{\bar{1}})$ . By the previous lemma, we can write  $\mathfrak{g}_{\bar{0}} = \mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}$  where  $X_{\bar{0}} = \phi^{-1}(\alpha)$ ,  $B(X_{\bar{0}}, X_{\bar{0}}) = B(Y_{\bar{0}}, Y_{\bar{0}}) = 0$ ,  $B(X_{\bar{0}}, Y_{\bar{0}}) = 1$ . Let  $X_{\bar{1}} = \phi^{-1}(p)$ ,  $Y_{\bar{1}} = \phi^{-1}(q)$  and  $U = \text{span}\{X_{\bar{1}}, Y_{\bar{1}}\}$ .

- (2) If  $\dim(U) = 1$  then  $Y_{\bar{1}} = kX_{\bar{1}}$  with some nonzero  $k \in \mathbb{C}$ . Therefore,  $q = kp$  and  $I = k\alpha \otimes p^2$ . Replacing  $X_{\bar{0}}$  by  $kX_{\bar{0}}$  and  $Y_{\bar{0}}$  by  $\frac{1}{k}Y_{\bar{0}}$ , we can assume that  $k = 1$ . Let  $Z_{\bar{1}}$  be an element in  $\mathfrak{g}_{\bar{1}}$  such that  $B(X_{\bar{1}}, Z_{\bar{1}}) = 1$ .

Now, let  $X \in \mathfrak{g}_{\bar{0}}, Y, Z \in \mathfrak{g}_{\bar{1}}$ . By using (1.7) and (1.8) of [BP89], one has:

$$B(X, [Y, Z]) = -2\alpha(X)p(Y)p(Z) = -2B(X_{\bar{0}}, X)B(X_{\bar{1}}, Y)B(X_{\bar{1}}, Z).$$

Since  $B|_{\mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{0}}}$  is non-degenerate then:

$$[Y, Z] = -2B(X_{\bar{1}}, Y)B(X_{\bar{1}}, Z)X_{\bar{0}}, \quad \forall Y, Z \in \mathfrak{g}_{\bar{1}}.$$

Similarly and by the invariance of  $B$ , we also obtain:

$$[X, Y] = -2B(X_{\bar{0}}, X)B(X_{\bar{1}}, Y)X_{\bar{1}}, \quad \forall X \in \mathfrak{g}_{\bar{0}}, Y \in \mathfrak{g}_{\bar{1}}$$

and (2) follows.

- (3) If  $\dim(U) = 2$  and  $U$  is non-degenerate then  $B(X_{\bar{1}}, Y_{\bar{1}}) = a \neq 0$ . Replacing  $X_{\bar{1}}$  by  $\frac{1}{a}X_{\bar{1}}$ ,  $X_{\bar{0}}$  by  $aX_{\bar{0}}$  and  $Y_{\bar{0}}$  by  $\frac{1}{a}Y_{\bar{0}}$ , we can assume that  $a = 1$ . Then one has  $\mathfrak{g}_{\bar{0}} = \text{span}\{X_{\bar{0}}, Y_{\bar{0}}\}$ ,  $\mathfrak{g}_{\bar{1}} = \text{span}\{X_{\bar{1}}, Y_{\bar{1}}\}$ ,  $B(X_{\bar{0}}, Y_{\bar{0}}) = B(X_{\bar{1}}, Y_{\bar{1}}) = 1$  and  $I = \alpha \otimes pq$ . Let  $X \in \mathfrak{g}_{\bar{0}}, Y, Z \in \mathfrak{g}_{\bar{1}}$ , we have:

$$B(X, [Y, Z]) = I(X, Y, Z) = -\alpha(X)(p(Y)q(Z) + p(Z)q(Y)).$$

Therefore, the Lie super-bracket is defined:

$$[Y, Z] = -(B(X_{\bar{1}}, Y)B(Y_{\bar{1}}, Z) + B(X_{\bar{1}}, Z)B(Y_{\bar{1}}, Y))X_{\bar{0}}, \quad \forall Y, Z \in \mathfrak{g}_{\bar{1}},$$

$$[X, Y] = -B(X_{\bar{0}}, X)(B(X_{\bar{1}}, Y)Y_{\bar{1}} + B(Y_{\bar{1}}, Y)X_{\bar{1}}), \quad \forall X \in \mathfrak{g}_{\bar{0}}, Y \in \mathfrak{g}_{\bar{1}}$$

and (3) follows.

- (4) If  $\dim(U) = 2$  and  $U$  is totally isotropic: let  $V = \text{span}\{Z_{\bar{1}}, T_{\bar{1}}\}$  be a 2-dimensional totally isotropic subspace of  $\mathfrak{g}_{\bar{1}}$  such that  $\mathfrak{g}_{\bar{1}} = U \oplus V$  and  $B(X_{\bar{1}}, Z_{\bar{1}}) = B(Y_{\bar{1}}, T_{\bar{1}}) = 1$ . If  $X \in \mathfrak{g}_{\bar{0}}, Y, Z \in \mathfrak{g}_{\bar{1}}$  then:

$$B(X, [Y, Z]) = I(X, Y, Z) = -\alpha(X)(p(Y)q(Z) + p(Z)q(Y)).$$

We obtain the Lie super-bracket as follows:

$$[Y, Z] = -(B(X_{\bar{1}}, Y)B(Y_{\bar{1}}, Z) + B(X_{\bar{1}}, Z)B(Y_{\bar{1}}, Y))X_{\bar{0}}, \quad \forall Y, Z \in \mathfrak{g}_{\bar{1}},$$

$$[X, Y] = -B(X_{\bar{0}}, X)(B(X_{\bar{1}}, Y)Y_{\bar{1}} + B(Y_{\bar{1}}, Y)X_{\bar{1}}), \quad \forall X \in \mathfrak{g}_{\bar{0}}, Y \in \mathfrak{g}_{\bar{1}}.$$

$$\text{Thus, } [Z_{\bar{1}}, T_{\bar{1}}] = -X_{\bar{0}}, [Y_{\bar{0}}, Z_{\bar{1}}] = -Y_{\bar{1}}, [Y_{\bar{0}}, T_{\bar{1}}] = -X_{\bar{1}}.$$

□

#### 4. QUADRATIC LIE SUPERALGEBRAS WITH 2-DIMENSIONAL EVEN PART

This section is devoted to study another particular case of singular quadratic Lie superalgebras: quadratic Lie superalgebras with 2-dimensional even part. As we shall see, they can be seen as a symplectic version of solvable singular quadratic Lie algebras. The first result classifies these algebras with respect to the dup-number.

**Proposition 4.1.** *Let  $\mathfrak{g}$  be a non-Abelian quadratic Lie superalgebra with  $\dim(\mathfrak{g}_{\bar{0}}) = 2$ . Then  $\mathfrak{g}$  is a singular quadratic Lie superalgebra of type  $S_1$ .*

*Proof.* Let  $I$  be the associated invariant of  $\mathfrak{g}$ . By a remark in [PU07], every non-Abelian quadratic Lie algebra must have the dimension more than 2 so  $\mathfrak{g}_{\bar{0}}$  is Abelian and as a consequence,  $I \in \text{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^2(\mathfrak{g}_{\bar{1}})$ . We choose a basis  $\{X_{\bar{0}}, Y_{\bar{0}}\}$  of  $\mathfrak{g}_{\bar{0}}$  such that  $B(X_{\bar{0}}, X_{\bar{0}}) = B(Y_{\bar{0}}, Y_{\bar{0}}) = 0$  and  $B(X_{\bar{0}}, Y_{\bar{0}}) = 1$ . Let  $\alpha = \phi(X_{\bar{0}})$ ,  $\beta = \phi(Y_{\bar{0}})$  and we can assume that

$$I = \alpha \otimes \Omega_1 + \beta \otimes \Omega_2$$

where  $\Omega_1, \Omega_2 \in \text{Sym}^2(\mathfrak{g}_{\bar{1}})$ . Then one has:

$$\{I, I\} = 2(\Omega_1\Omega_2 + \alpha \wedge \beta \otimes \{\Omega_1, \Omega_2\}).$$

Therefore,  $\{I, I\} = 0$  implies that  $\Omega_1\Omega_2 = 0$ . So  $\Omega_1 = 0$  or  $\Omega_2 = 0$ . It means that  $\mathfrak{g}$  is a singular quadratic Lie superalgebra of type  $S_1$ . □

**Proposition 4.2.** *Let  $\mathfrak{g}$  be a singular quadratic Lie superalgebra with Abelian even part. If  $\mathfrak{g}$  is reduced then  $\dim(\mathfrak{g}_{\bar{0}}) = 2$ .*

*Proof.* Let  $I$  be the associated invariant of  $\mathfrak{g}$ . Since  $\mathfrak{g}$  has the Abelian even part one has  $I \in \text{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^2(\mathfrak{g}_{\bar{1}})$ . Moreover  $\mathfrak{g}$  is singular then

$$I = \alpha \otimes \Omega$$

where  $\alpha \in \mathfrak{g}_{\bar{0}}^*$ ,  $\Omega \in \text{Sym}^2(\mathfrak{g}_{\bar{1}})$ . The proof follows exactly Lemma 3.2. Let  $\beta \in \mathfrak{g}_{\bar{0}}^*$  then  $\{\beta, \alpha\} = 0$  if and only if  $\{\beta, I\} = 0$ , equivalently  $\phi^{-1}(\beta) \in \mathcal{Z}(\mathfrak{g})$ . Therefore,  $(\phi^{-1}(\alpha))^\perp \cap \mathfrak{g}_{\bar{0}} \subset \mathcal{Z}(\mathfrak{g})$ . It means that  $\dim(\mathfrak{g}_{\bar{0}}) = 2$  since  $\mathfrak{g}$  is reduced and  $\phi^{-1}(\alpha)$  is isotropic in  $\mathcal{Z}(\mathfrak{g})$ .  $\square$

Now, let  $\mathfrak{g}$  be a non-Abelian quadratic Lie superalgebra with 2-dimensional even part. By Proposition 4.1,  $\mathfrak{g}$  is singular of type  $S_1$ . Fix  $\alpha \in \mathcal{V}_I$  and choose  $\Omega \in \text{Sym}^2(\mathfrak{g})$  such that

$$I = \alpha \otimes \Omega.$$

We define  $C : \mathfrak{g}_{\bar{1}} \rightarrow \mathfrak{g}_{\bar{1}}$  by  $B(C(X), Y) = \Omega(X, Y)$ , for all  $X, Y \in \mathfrak{g}_{\bar{1}}$  and let  $X_{\bar{0}} = \phi^{-1}(\alpha)$ .

**Lemma 4.3.** *The following assertions are equivalent:*

- (1)  $\{I, I\} = 0$ ,
- (2)  $\{\alpha, \alpha\} = 0$ ,
- (3)  $B(X_{\bar{0}}, X_{\bar{0}}) = 0$ .

*In this case, one has  $X_{\bar{0}} \in \mathcal{Z}(\mathfrak{g})$ .*

*Proof.* It is easy to see that:

$$\{I, I\} = 0 \Leftrightarrow \{\alpha, \alpha\} \otimes \Omega^2 = 0.$$

Therefore the assertions are equivalent. Moreover, since  $\{\alpha, I\} = 0$  one has  $X_{\bar{0}} \in \mathcal{Z}(\mathfrak{g})$ .  $\square$

We keep the notations as in the previous sections. Then there exists an isotropic element  $Y_{\bar{0}} \in \mathfrak{g}_{\bar{0}}$  such that  $B(X_{\bar{0}}, Y_{\bar{0}}) = 1$  and one has the following proposition:

**Proposition 4.4.**

- (1) *The map  $C$  is skew-symmetric (with respect to  $B$ ), that is*

$$B(C(X), Y) = -B(X, C(Y))$$

*for all  $X, Y \in \mathfrak{g}_{\bar{1}}$ .*

- (2)  $[X, Y] = B(C(X), Y)X_{\bar{0}}$ , for all  $X, Y \in \mathfrak{g}_{\bar{1}}$  and  $C = \text{ad}(Y_{\bar{0}})|_{\mathfrak{g}_{\bar{1}}}$ .
- (3)  $\mathcal{Z}(\mathfrak{g}) = \ker(C) \oplus \mathbb{C}X_{\bar{0}}$  and  $[\mathfrak{g}, \mathfrak{g}] = \text{Im}(C) \oplus \mathbb{C}X_{\bar{0}}$ . Therefore,  $\mathfrak{g}$  is reduced if and only if  $\ker(C) \subset \text{Im}(C)$ .
- (4) *The Lie superalgebra  $\mathfrak{g}$  is solvable. Moreover,  $\mathfrak{g}$  is nilpotent if and only if  $C$  is nilpotent.*

*Proof.*

- (1) For all  $X, Y \in \mathfrak{g}_{\bar{1}}$ , one has

$$B(C(X), Y) = \Omega(X, Y) = \Omega(Y, X) = B(C(Y), X) = -B(X, C(Y)).$$

- (2) Let  $X \in \mathfrak{g}_{\bar{0}}, Y, Z \in \mathfrak{g}_{\bar{1}}$  then

$$B(X, [Y, Z]) = (\alpha \otimes \Omega)(X, Y, Z) = \alpha(X)\Omega(Y, Z).$$

Since  $\alpha(X) = B(X_{\bar{0}}, X)$  and  $\Omega(Y, Z) = B(C(Y), Z)$  so one has

$$B(X, [Y, Z]) = B(X_{\bar{0}}, X)B(C(Y), Z).$$

The non-degeneracy of  $B$  implies  $[Y, Z] = B(C(Y), Z)X_{\bar{0}}$ . Set  $X = Y_{\bar{0}}$  then  $B(Y_{\bar{0}}, [Y, Z]) = B(C(Y), Z)$ . By the invariance of  $B$ , we obtain  $[Y_{\bar{0}}, Y] = C(Y)$ .

- (3) It follows from the assertion (2).  
 (4)  $\mathfrak{g}$  is solvable since  $\mathfrak{g}_{\bar{0}}$  is solvable, or since  $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \subset \mathbb{C}X_{\bar{0}}$ . If  $\mathfrak{g}$  is nilpotent then  $C = \text{ad}(Y_{\bar{0}})$  is nilpotent obviously. Conversely, if  $C$  is nilpotent then it is easy to see that  $\mathfrak{g}$  is nilpotent since  $(\text{ad}(X))^k(\mathfrak{g}) \subset \mathbb{C}X_{\bar{0}} \oplus \text{Im}(C^k)$  for all  $X \in \mathfrak{g}$ .

□

*Remark 4.5.* The choice of  $C$  is unique up to a nonzero scalar. Indeed, assume that  $I = \alpha' \otimes \Omega'$  and  $C'$  is the map associated to  $\Omega'$ . Since  $\mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} = \mathbb{C}X_{\bar{0}}$  and  $\phi^{-1}(\alpha') \in \mathcal{Z}(\mathfrak{g})$  one has  $\alpha' = \lambda\alpha$  for some nonzero  $\lambda \in \mathbb{C}$ . Therefore,  $\alpha \otimes (\Omega - \lambda\Omega') = 0$ . It means that  $\Omega = \lambda\Omega'$  and then we get  $C = \lambda C'$ .

#### 4.1. Double extension of a symplectic vector space.

Double extensions are a very useful method initiated by V. Kac to construct quadratic Lie algebras (see [Kac85] and [MR85]). They are generalized to many algebras endowed with a non-degenerate invariant bilinear form, for example quadratic Lie superalgebras (see [BB99] and [BBB]). In [DPU], we consider a particular case that is the double extension of a quadratic vector space by a skew-symmetric map. From this we obtain the class of solvable singular quadratic Lie algebras. Here, we use this notion in yet another context, replacing the quadratic vector space by a symplectic vector space.

##### Definition 4.6.

- (1) Let  $(\mathfrak{q}, B_{\mathfrak{q}})$  be a symplectic vector space equipped with a symplectic bilinear form  $B_{\mathfrak{q}}$  and  $\bar{C} : \mathfrak{q} \rightarrow \mathfrak{q}$  be a skew-symmetric map, that is,

$$B_{\mathfrak{q}}(\bar{C}(X), Y) = -B_{\mathfrak{q}}(X, \bar{C}(Y)), \forall X, Y \in \mathfrak{q}.$$

Let  $(\mathfrak{t} = \text{span}\{X_{\bar{0}}, Y_{\bar{0}}\}, B_{\mathfrak{t}})$  be a 2-dimensional quadratic vector space with the symmetric bilinear form  $B_{\mathfrak{t}}$  defined by

$$B_{\mathfrak{t}}(X_{\bar{0}}, X_{\bar{0}}) = B_{\mathfrak{t}}(Y_{\bar{0}}, Y_{\bar{0}}) = 0, B_{\mathfrak{t}}(X_{\bar{0}}, Y_{\bar{0}}) = 1.$$

Consider the vector space

$$\mathfrak{g} = \mathfrak{t} \oplus^{\perp} \mathfrak{q}$$

equipped with the bilinear form  $B = B_{\mathfrak{t}} + B_{\mathfrak{q}}$  and define a bracket on  $\mathfrak{g}$  by

$$[\lambda X_{\bar{0}} + \mu Y_{\bar{0}} + X, \lambda' X_{\bar{0}} + \mu' Y_{\bar{0}} + Y] = \mu \bar{C}(Y) - \mu' \bar{C}(X) + B(\bar{C}(X), Y)X_{\bar{0}},$$



for all  $X, Y \in \mathfrak{q}, \lambda, \mu, \lambda', \mu' \in \mathbb{C}$ . Then  $(\mathfrak{g}, B)$  is a quadratic solvable Lie superalgebra with  $\mathfrak{g}_{\bar{0}} = \mathfrak{t}$  and  $\mathfrak{g}_{\bar{1}} = \mathfrak{q}$ . We say that  $\mathfrak{g}$  is the *double extension of  $\mathfrak{q}$  by  $\bar{C}$* .

- (2) Let  $\mathfrak{g}_i$  be double extensions of symplectic vector spaces  $(\mathfrak{q}_i, B_i)$  by skew-symmetric maps  $\bar{C}_i \in \mathcal{L}(\mathfrak{q}_i)$ , for  $1 \leq i \leq k$ . The *amalgamated product*

$$\mathfrak{g} = \mathfrak{g}_1 \times_a \mathfrak{g}_2 \times_a \dots \times_a \mathfrak{g}_k$$

is defined as follows:

- consider  $(\mathfrak{q}, B)$  be the symplectic vector space with  $\mathfrak{q} = \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \dots \oplus \mathfrak{q}_k$  and the bilinear form  $B$  such that  $B(\sum_{i=1}^k X_i, \sum_{i=1}^k Y_i) = \sum_{i=1}^k B_i(X_i, Y_i)$ , for  $X_i, Y_i \in \mathfrak{q}_i$ ,  $1 \leq i \leq k$ .
- the skew-symmetric map  $\bar{C} \in \mathcal{L}(\mathfrak{q})$  is defined by  $\bar{C}(\sum_{i=1}^k X_i) = \sum_{i=1}^k \bar{C}_i(X_i)$ , for  $X_i \in \mathfrak{q}_i$ ,  $1 \leq i \leq k$ .

Then  $\mathfrak{g}$  is the double extension of  $\mathfrak{q}$  by  $\bar{C}$ .

**Lemma 4.7.** *We keep the notation above.*

- (1) *Let  $\mathfrak{g}$  be the double extension of  $\mathfrak{q}$  by  $\bar{C}$ . Then*

$$[X, Y] = B(X_{\bar{0}}, Y)C(Y) - B(X_{\bar{0}}, Y)C(X) + B(C(X), Y)X_{\bar{0}}, \quad \forall X, Y \in \mathfrak{g},$$

*where  $C = \text{ad}(Y_{\bar{0}})$ . Moreover,  $X_{\bar{0}} \in \mathcal{Z}(\mathfrak{g})$  and  $C|_{\mathfrak{q}} = \bar{C}$ .*

- (2) *Let  $\mathfrak{g}'$  be the double extension of  $\mathfrak{q}$  by  $\bar{C}' = \lambda \bar{C}$ ,  $\lambda \in \mathbb{C}$ ,  $\lambda \neq 0$ . Then  $\mathfrak{g}$  and  $\mathfrak{g}'$  are i-isomorphic.*

*Proof.*

- (1) This is a straightforward computation by Definition 4.6.

- (2) Write  $\mathfrak{g} = \mathfrak{t} \oplus^{\perp} \mathfrak{q} = \mathfrak{g}'$ . Denote by  $[\cdot, \cdot]'$  the Lie super-bracket on  $\mathfrak{g}'$ . Define  $A : \mathfrak{g} \rightarrow \mathfrak{g}'$  by  $A(X_{\bar{0}}) = \lambda X_{\bar{0}}$ ,  $A(Y_{\bar{0}}) = \frac{1}{\lambda} Y_{\bar{0}}$  and  $A|_{\mathfrak{q}} = \text{Id}_{\mathfrak{q}}$ . Then  $A([Y_{\bar{0}}, X]) = C(X) = [A(Y_{\bar{0}}), A(X)]'$  and  $A([X, Y]) = [A(X), A(Y)]'$ , for all  $X, Y \in \mathfrak{q}$ . So  $A$  is an i-isomorphism. □

**Proposition 4.8.**

- (1) *Let  $\mathfrak{g}$  be a non-Abelian quadratic Lie superalgebra with 2-dimensional even part. Keep the notations as in Proposition 4.4. Then  $\mathfrak{g}$  is the double extension of  $\mathfrak{q} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}})^{\perp} = \mathfrak{g}_{\bar{1}}$  by  $\bar{C} = \text{ad}(Y_{\bar{0}})|_{\mathfrak{q}}$ .*
- (2) *Let  $\mathfrak{g}$  be the double extension of a symplectic vector space  $\mathfrak{q}$  by a map  $\bar{C} \neq 0$ . Then  $\mathfrak{g}$  is a singular solvable quadratic Lie superalgebra with 2-dimensional even part. Moreover:*
- (i)  *$\mathfrak{g}$  is reduced if and only if  $\ker(\bar{C}) \subset \text{Im}(\bar{C})$ .*
  - (ii)  *$\mathfrak{g}$  is nilpotent if and only if  $\bar{C}$  is nilpotent.*
- (3) *Let  $(\mathfrak{g}, B)$  be a quadratic Lie superalgebra. Let  $\mathfrak{g}'$  be the double extension of a symplectic vector space  $(\mathfrak{q}', B')$  by a map  $\bar{C}'$ . Let  $A$  be an i-isomorphism of  $\mathfrak{g}'$  onto  $\mathfrak{g}$  and write  $\mathfrak{q} = A(\mathfrak{q}')$ . Then  $\mathfrak{g}$  is the double extension of  $(\mathfrak{q}, B|_{\mathfrak{q} \times \mathfrak{q}})$  by the map  $\bar{C} = \bar{A} \bar{C}' \bar{A}^{-1}$  where  $\bar{A} = A|_{\mathfrak{q}'}$ .*

*Proof.* The assertions (1) and (2) follow Proposition 4.4 and Lemma 4.7. For (3), since  $A$  is  $i$ -isomorphic then  $\mathfrak{g}$  has also 2-dimensional even part. Write  $\mathfrak{g}' = (\mathbb{C}X'_0 \oplus \mathbb{C}Y'_0) \oplus \mathfrak{q}'$ . Let  $X_0 = A(X'_0)$  and  $Y_0 = A(Y'_0)$ . Then  $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \oplus \mathfrak{q}$  and one has:

$$[Y_0, X] = (A\overline{C'}A^{-1})(X), \forall X \in \mathfrak{q}, \text{ and}$$

$$[X, Y] = B((A\overline{C'}A^{-1})(X), Y)X_0, \forall X, Y \in \mathfrak{q}.$$

This proves the result.  $\square$

**Example 4.9.** From the point of view of double extensions, for reduced elementary quadratic Lie superalgebras with 2-dimensional even part in Section 3 one has

- (1)  $\mathfrak{g}_{4,1}^s$  is the double extension of the 2-dimensional symplectic vector space  $\mathfrak{q} = \mathbb{C}^2$  by the map having matrix:

$$\overline{C} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in a Darboux basis  $\{E_1, E_2\}$  of  $\mathfrak{q}$  where  $B(E_1, E_2) = 1$ .

- (2)  $\mathfrak{g}_{4,2}^s$  the double extension of the 2-dimensional symplectic vector space  $\mathfrak{q} = \mathbb{C}^2$  by the map having matrix:

$$\overline{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in a Darboux basis  $\{E_1, E_2\}$  of  $\mathfrak{q}$  where  $B(E_1, E_2) = 1$ .

- (3)  $\mathfrak{g}_6^s$  is the double extension of the 4-dimensional symplectic vector space  $\mathfrak{q} = \mathbb{C}^4$  by the map having matrix:

$$\overline{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

in a Darboux basis  $\{E_1, E_2, E_3, E_4\}$  of  $\mathfrak{q}$  where  $B(E_1, E_3) = B(E_2, E_4) = 1$ , the other are zero.

Let  $(\mathfrak{q}, B)$  be a symplectic vector space. We denote by  $\text{Sp}(\mathfrak{q})$  the isometry group of the symplectic form  $B$  and by  $\mathfrak{sp}(\mathfrak{q})$  its Lie algebra, i.e. the Lie algebra of skew-symmetric maps with respects to  $B$ . The *adjoint action* is the action of  $\text{Sp}(\mathfrak{q})$  on  $\mathfrak{sp}(\mathfrak{q})$  by the conjugation (see Appendix). Also, we denote by  $\mathbb{P}^1(\mathfrak{sp}(2n))$  the projective space of  $\mathfrak{sp}(2n)$  with the action induced by  $\text{Sp}(2n)$ -adjoint action on  $\mathfrak{sp}(2n)$ .

**Proposition 4.10.** *Let  $(\mathfrak{q}, B)$  be a symplectic vector space. Let  $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \oplus \mathfrak{q}$  and  $\mathfrak{g}' = (\mathbb{C}X'_0 \oplus \mathbb{C}Y'_0) \oplus \mathfrak{q}$  be double extensions of  $\mathfrak{q}$ , by skew-symmetric maps  $\overline{C}$  and  $\overline{C'}$  respectively. Then:*

- (1) *there exists a Lie superalgebra isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}'$  if and only if there exist an invertible map  $P \in \mathcal{L}(\mathfrak{q})$  and a nonzero  $\lambda \in \mathbb{C}$  such that*

$\overline{C'} = \lambda \overline{P} \overline{C} P^{-1}$  and  $P^* \overline{P} \overline{C} = \overline{C}$  where  $P^*$  is the adjoint map of  $P$  with respect to  $B$ .

- (2) there exists an  $i$ -isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}'$  if and only if  $\overline{C'}$  is in the  $\text{Sp}(\mathfrak{q})$ -adjoint orbit through  $\lambda \overline{C}$  for some nonzero  $\lambda \in \mathbb{C}$ .

*Proof.* The assertions are obvious if  $\overline{C} = 0$ . We assume  $\overline{C} \neq 0$ .

- (1) Let  $A : \mathfrak{g} \rightarrow \mathfrak{g}'$  be a Lie superalgebra isomorphism then  $A(\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) = \mathbb{C}X'_{\overline{0}} \oplus \mathbb{C}Y'_{\overline{0}}$  and  $A(\mathfrak{q}) = \mathfrak{q}$ . It is obvious that  $\overline{C'} \neq 0$ . It is easy to see that  $\mathbb{C}X_{\overline{0}} = \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\overline{0}}$  and  $\mathbb{C}X'_{\overline{0}} = \mathcal{Z}(\mathfrak{g}') \cap \mathfrak{g}'_{\overline{0}}$  then one has  $A(\mathbb{C}X_{\overline{0}}) = \mathbb{C}X'_{\overline{0}}$ . It means  $A(X_{\overline{0}}) = \mu X'_{\overline{0}}$  for some nonzero  $\mu \in \mathbb{C}$ . Let  $A|_{\mathfrak{q}} = Q$  and assume  $A(Y_{\overline{0}}) = \beta Y'_{\overline{0}} + \gamma X'_{\overline{0}}$ . For all  $X, Y \in \mathfrak{q}$ , we have  $A([X, Y]) = \mu B(\overline{C}(X), Y) X'_{\overline{0}}$ . Also,

$$A([X, Y]) = [Q(X), Q(Y)]' = B(\overline{C'}Q(X), Q(Y))X'_{\overline{0}}.$$

It results that  $Q^* \overline{C'} Q = \mu \overline{C}$ .

Moreover,  $A([Y_{\overline{0}}, X]) = Q(\overline{C}(X)) = [\beta Y'_{\overline{0}} + \gamma X'_{\overline{0}}, Q(X)]' = \beta \overline{C'} Q(X)$ , for all  $X \in \mathfrak{q}$ . We conclude that  $Q \overline{C} Q^{-1} = \beta \overline{C'}$  and since  $Q^* \overline{C'} Q = \mu \overline{C}$ , then  $Q^* Q \overline{C} = \beta \mu \overline{C}$ .

Set  $P = \frac{1}{(\mu\beta)^{\frac{1}{2}}} Q$  and  $\lambda = \frac{1}{\beta}$ . It follows that  $\overline{C'} = \lambda \overline{P} \overline{C} P^{-1}$  and  $P^* \overline{P} \overline{C} = \overline{C}$ .

Conversely, assume that  $\mathfrak{g} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$  and  $\mathfrak{g}' = (\mathbb{C}X'_{\overline{0}} \oplus \mathbb{C}Y'_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$  be double extensions of  $\mathfrak{q}$ , by skew-symmetric maps  $\overline{C}$  and  $\overline{C'}$  respectively such that  $\overline{C'} = \lambda \overline{P} \overline{C} P^{-1}$  and  $P^* \overline{P} \overline{C} = \overline{C}$  with an invertible map  $P \in \mathcal{L}(\mathfrak{q})$  and a nonzero  $\lambda \in \mathbb{C}$ . Define  $A : \mathfrak{g} \rightarrow \mathfrak{g}'$  by  $A(X_{\overline{0}}) = \lambda X'_{\overline{0}}$ ,  $A(Y_{\overline{0}}) = \frac{1}{\lambda} Y'_{\overline{0}}$  and  $A(X) = P(X)$ , for all  $X \in \mathfrak{q}$  then it is easy to check that  $A$  is a Lie superalgebra isomorphism.

- (2) If  $\mathfrak{g}$  and  $\mathfrak{g}'$  are  $i$ -isomorphic, then the isomorphism  $A$  in the proof of (1) is an isometry. Hence  $P \in \text{Sp}(\mathfrak{q})$  and  $\overline{C'} = \lambda \overline{P} \overline{C} P^{-1}$  gives the result.

Conversely, define  $A$  as above (the sufficiency of (1)). Then  $A$  is an isometry and it is easy to check that  $A$  is an  $i$ -isomorphism.

□

**Corollary 4.11.** Let  $(\mathfrak{g}, B)$  and  $(\mathfrak{g}', B')$  be double extensions of  $(\mathfrak{q}, \overline{B})$  and  $(\mathfrak{q}', \overline{B'})$  respectively where  $\overline{B} = B|_{\mathfrak{q} \times \mathfrak{q}}$  and  $\overline{B'} = B'|_{\mathfrak{q}' \times \mathfrak{q}'}$ . Write  $\mathfrak{g} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$  and  $\mathfrak{g}' = (\mathbb{C}X'_{\overline{0}} \oplus \mathbb{C}Y'_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}'$ . Then:

- (1) there exists an  $i$ -isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}'$  if and only if there exists an isometry  $\overline{A} : \mathfrak{q} \rightarrow \mathfrak{q}'$  such that  $\overline{C'} = \lambda \overline{A} \overline{C} \overline{A}^{-1}$ , for some nonzero  $\lambda \in \mathbb{C}$ .
- (2) there exists a Lie superalgebra isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}'$  if and only if there exist invertible maps  $\overline{Q} : \mathfrak{q} \rightarrow \mathfrak{q}'$  and  $\overline{P} \in \mathcal{L}(\mathfrak{q})$  such that
- (i)  $\overline{C'} = \lambda \overline{Q} \overline{C} \overline{Q}^{-1}$  for some nonzero  $\lambda \in \mathbb{C}$ ,
  - (ii)  $\overline{P}^* \overline{P} \overline{C} = \overline{C}$  and

(iii)  $\overline{Q} \overline{P}^{-1}$  is an isometry from  $\mathfrak{q}$  onto  $\mathfrak{q}'$ .

*Proof.* The proof is completely similar to Corollary 4.6 in [DPU]. It is sketched as follows. First we can assume  $\dim(\mathfrak{g}) = \dim(\mathfrak{g}')$  and define then a map  $F : \mathfrak{g}' \rightarrow \mathfrak{g}$  by  $F(X'_0) = X_0$ ,  $F(Y'_0) = Y_0$  and  $\bar{F} = F|_{\mathfrak{q}'}$  is an isometry from  $\mathfrak{q}'$  onto  $\mathfrak{q}$ . We define a new Lie bracket on  $\mathfrak{g}$  by

$$[X, Y]'' = F([F^{-1}(X), F^{-1}(Y)]'), \forall X, Y \in \mathfrak{g}.$$

and denote by  $(\mathfrak{g}'', [\cdot, \cdot]'')$  this new Lie superalgebra. So  $F$  is an i-isomorphism from  $\mathfrak{g}'$  onto  $\mathfrak{g}''$  and by Proposition 4.8 (3)  $\mathfrak{g}'' = (\mathbb{C}X_1 \oplus \mathbb{C}Y_1) \overset{\perp}{\oplus} \mathfrak{q}$  is the double extension of  $\mathfrak{q}$  by  $\overline{C}''$  with  $\overline{C}'' = \overline{F} \overline{C}' \overline{F}^{-1}$ . We have that  $\mathfrak{g}$  and  $\mathfrak{g}'$  are isomorphic (resp. i-isomorphic) if and only if  $\mathfrak{g}$  and  $\mathfrak{g}''$  are isomorphic (resp. i-isomorphic). Finally, by applying Proposition 4.11 to the Lie superalgebras  $\mathfrak{g}$  and  $\mathfrak{g}''$  we obtain the corollary.  $\square$

It results that quadratic Lie superalgebra structures on the quadratic  $\mathbb{Z}_2$ -graded vector space  $\mathbb{C}^2 \overset{\mathbb{Z}_2}{\oplus} \mathbb{C}^{2n}$  can be classified up to i-isomorphism in terms of  $\text{Sp}(2n)$ -orbits in  $\mathbb{P}^1(\mathfrak{sp}(2n))$ . This work is like what have been done in [DPU]. We need the following lemma:

**Lemma 4.12.** *Let  $V$  be a quadratic  $\mathbb{Z}_2$ -graded vector space such that its even part is 2-dimensional. We write  $V = (\mathbb{C}X'_0 \oplus \mathbb{C}Y'_0) \overset{\perp}{\oplus} \mathfrak{q}'$  with  $X'_0, Y'_0$  isotropic elements in  $V_0$  and  $B(X'_0, Y'_0) = 1$ . Let  $\mathfrak{g}$  be a quadratic Lie superalgebra with  $\dim(\mathfrak{g}_0) = \dim(V_0)$  and  $\dim(\mathfrak{g}) = \dim(V)$ . Then, there exists a skew-symmetric map  $\overline{C}' : \mathfrak{q}' \rightarrow \mathfrak{q}'$  such that  $V$  is considered as the double extension of  $\mathfrak{q}'$  by  $\overline{C}'$  that is i-isomorphic to  $\mathfrak{g}$ .*

*Proof.* By Proposition 4.8,  $\mathfrak{g}$  is a double extension. Let us write  $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \overset{\perp}{\oplus} \mathfrak{q}$  and  $\overline{C} = \text{ad}(Y_0)|_{\mathfrak{q}}$ . Define  $A : \mathfrak{g} \rightarrow V$  by  $A(X_0) = X'_0$ ,  $A(Y_0) = Y'_0$  and  $\overline{A} = A|_{\mathfrak{q}}$  any isometry from  $\mathfrak{q} \rightarrow \mathfrak{q}'$ . It is clear that  $A$  is an isometry from  $\mathfrak{g}$  to  $V$ . Now, define the Lie super-bracket on  $V$  by:

$$[X, Y] = A([A^{-1}(X), A^{-1}(Y)]), \forall X, Y \in V.$$

Then  $V$  is a quadratic Lie superalgebra, that is i-isomorphic to  $\mathfrak{g}$ . Moreover,  $V$  is obviously the double extension of  $\mathfrak{q}'$  by  $\overline{C}' = \overline{A} \overline{C} \overline{A}^{-1}$ .  $\square$

Proposition 4.8, Proposition 4.10, Corollary 4.11 and Lemma 4.12 are enough for us to apply the classification method in [DPU] for the set  $\mathcal{S}(2+2n)$  of quadratic Lie superalgebra structures on the quadratic  $\mathbb{Z}_2$ -graded vector space  $\mathbb{C}^2 \overset{\mathbb{Z}_2}{\oplus} \mathbb{C}^{2n}$  by only replacing the isometry group  $\text{O}(m)$  by  $\text{Sp}(2n)$  and  $\mathfrak{o}(m)$  by  $\mathfrak{sp}(2n)$  to obtain completely similar results. One has the first characterization of the set  $\mathcal{S}(2+2n)$ :

**Proposition 4.13.** *Let  $\mathfrak{g}$  and  $\mathfrak{g}'$  be elements in  $\mathcal{S}(2+2n)$ . Then  $\mathfrak{g}$  and  $\mathfrak{g}'$  are i-isomorphic if and only if they are isomorphic.*

By using the notion of double extension, we call the Lie superalgebra  $\mathfrak{g} \in \mathcal{S}(2+2n)$  *diagonalizable* (resp. *invertible*) if it is a double extension by a diagonalizable

(resp. invertible) map. Denote the subsets of nilpotent elements, diagonalizable elements and invertible elements in  $\mathcal{S}(2+2n)$ , respectively by  $\mathcal{N}(2+2n)$ ,  $\mathcal{D}(2+2n)$  and by  $\mathcal{S}_{\text{inv}}(2+2n)$ . Denote by  $\widehat{\mathcal{N}}(2+2n)$ ,  $\widehat{\mathcal{D}}(2+2n)$ ,  $\widehat{\mathcal{S}_{\text{inv}}}(2+2n)$  the sets of isomorphism classes in  $\mathcal{N}(2+2n)$ ,  $\mathcal{D}(2+2n)$ ,  $\mathcal{S}_{\text{inv}}(2+2n)$ , respectively and  $\widehat{\mathcal{D}}_{\text{red}}(2+2n)$  the subset of  $\widehat{\mathcal{D}}(2+2n)$  including reduced ones. Applying Appendix, we have the classification result of these sets as follows:

**Proposition 4.14.**

- (1) *There is a bijection between  $\widehat{\mathcal{N}}(2+2n)$  and the set of nilpotent  $\text{Sp}(2n)$ -adjoint orbits of  $\mathfrak{sp}(2n)$  that induces a bijection between  $\widehat{\mathcal{N}}(2+2n)$  and the set of partitions  $\mathcal{P}_{-1}(2n)$ .*
- (2) *There is a bijection between  $\widehat{\mathcal{D}}(2+2n)$  and the set of semisimple  $\text{Sp}(2n)$ -orbits of  $\mathbb{P}^1(\mathfrak{sp}(2n))$  that induces a bijection between  $\widehat{\mathcal{D}}(2+2n)$  and  $\Lambda_n/H_n$  where  $H_n$  is the group obtained from the group  $G_n$  by adding maps  $(\lambda_1, \dots, \lambda_n) \mapsto \lambda(\lambda_1, \dots, \lambda_n)$ ,  $\forall \lambda \in \mathbb{C}$ ,  $\lambda \neq 0$ . In the reduced case,  $\widehat{\mathcal{D}}_{\text{red}}(2+2n)$  is bijective to  $\Lambda_n^+/H_n$  with  $\Lambda_n^+ = \{(\lambda_1, \dots, \lambda_n) \mid \lambda_1, \dots, \lambda_n \in \mathbb{C}, \lambda_i \neq 0, \forall i\}$ .*
- (3) *There is a bijection between  $\widehat{\mathcal{S}_{\text{inv}}}(2+2n)$  and the set of invertible  $\text{Sp}(2n)$ -orbits of  $\mathbb{P}^1(\mathfrak{sp}(2n))$  that induces a bijection between  $\widehat{\mathcal{S}_{\text{inv}}}(2+2n)$  and  $\mathcal{I}_n/\mathbb{C}^*$ .*
- (4) *There is a bijection between  $\widehat{\mathcal{S}}(2+2n)$  and the set of  $\text{Sp}(2n)$ -orbits of  $\mathbb{P}^1(\mathfrak{sp}(2n))$  that induces a bijection between  $\widehat{\mathcal{S}}(2+2n)$  and  $\mathcal{D}(2n)/\mathbb{C}^*$ .*

Next, we will describe the sets  $\mathcal{N}(2+2n)$ ,  $\mathcal{D}_{\text{red}}(2+2n)$  the subset of  $\mathcal{D}(2+2n)$  including reduced ones, and  $\mathcal{S}_{\text{inv}}(2+2n)$  in term of amalgamated product in Definition 4.6. Remark that except for the nilpotent case, the amalgamated product may have a bad behavior with respect to isomorphisms.

**Definition 4.15.** Keep the notation  $J_p$  for the Jordan block of size  $p$  as in Appendix and define two types of double extension as follows:

- for  $p \geq 2$ , we consider the symplectic vector space  $\mathfrak{q} = \mathbb{C}^{2p}$  equipped with its canonical bilinear form  $\overline{B}$  and the map  $\overline{C}_{2p}^J$  having matrix

$$\begin{pmatrix} J_p & 0 \\ 0 & -{}^t J_p \end{pmatrix}$$

in a Darboux basis. Then  $\overline{C}_{2p}^J \in \mathfrak{sp}(2p)$  and we denote by  $j_{2p}$  the double extension of  $\mathfrak{q}$  by  $\overline{C}_{2p}^J$ . So  $j_{2p} \in \mathcal{N}(2+2p)$ .

- for  $p \geq 1$ , we consider the symplectic vector space  $\mathfrak{q} = \mathbb{C}^{2p}$  equipped with its canonical bilinear form  $\overline{B}$  and the map  $\overline{C}_{p+p}^J$  with matrix

$$\begin{pmatrix} J_p & M \\ 0 & -{}^t J_p \end{pmatrix}$$

in a Darboux basis where  $M = (m_{ij})$  denotes the  $p \times p$ -matrix with  $m_{p,p} = 1$  and  $m_{ij} = 0$  otherwise. Then  $\overline{C}_{p+p}^J \in \mathfrak{sp}(2p)$  and we denote by  $\mathfrak{j}_{p+p}$  the double extension of  $\mathfrak{q}$  by  $\overline{C}_{p+p}^J$ . So  $\mathfrak{j}_{p+p} \in \mathcal{N}(2+2p)$ .

The Lie superalgebras  $\mathfrak{j}_{2p}$  or  $\mathfrak{j}_{p+p}$  will be called *nilpotent Jordan-type Lie superalgebras*.

Keep the notations as in Appendix. For  $n \in \mathbb{N}$ ,  $n \neq 0$ , each  $[d] \in \mathcal{P}_{-1}(2n)$  can be written as

$$[d] = (p_1, p_1, p_2, p_2, \dots, p_k, p_k, 2q_1, \dots, 2q_\ell),$$

with all  $p_i$  odd,  $p_1 \geq p_2 \geq \dots \geq p_k$  and  $q_1 \geq q_2 \geq \dots \geq q_\ell$ .

We associate the partition  $[d]$  with the map  $\overline{C}_{[d]} \in \mathfrak{sp}(2n)$  having matrix

$$\text{diag}_{\mathfrak{g}_{k+\ell}}(\overline{C}_{2p_1}^J, \overline{C}_{2p_2}^J, \dots, \overline{C}_{2p_k}^J, \overline{C}_{q_1+q_1}^J, \dots, \overline{C}_{q_\ell+q_\ell}^J)$$

in a Darboux basis of  $\mathbb{C}^{2n}$  and denote by  $\mathfrak{g}_{[d]}$  the double extension of  $\mathbb{C}^{2n}$  by  $\overline{C}_{[d]}$ . Then  $\mathfrak{g}_{[d]} \in \mathcal{N}(2+2n)$  and  $\mathfrak{g}_{[d]}$  is an amalgamated product of nilpotent Jordan-type Lie superalgebras. More precisely,

$$\mathfrak{g}_{[d]} = \mathfrak{j}_{2p_1} \times_{\mathfrak{a}} \mathfrak{j}_{2p_2} \times_{\mathfrak{a}} \dots \times_{\mathfrak{a}} \mathfrak{j}_{2p_k} \times_{\mathfrak{a}} \mathfrak{j}_{q_1+q_1} \times_{\mathfrak{a}} \dots \times_{\mathfrak{a}} \mathfrak{j}_{q_\ell+q_\ell}.$$

**Proposition 4.16.** *Each  $\mathfrak{g} \in \mathcal{N}(2+2n)$  is  $i$ -isomorphic to a unique amalgamated product  $\mathfrak{g}_{[d]}$ ,  $[d] \in \mathcal{P}_{-1}(2n)$ , of nilpotent Jordan-type Lie superalgebras.*

For the reduced diagonalizable case, let  $\mathfrak{g}_4^s(\lambda)$  be the double extension of  $\mathfrak{q} = \mathbb{C}^2$  by  $\overline{C} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ ,  $\lambda \neq 0$ . By Lemma 4.7,  $\mathfrak{g}_4^s(\lambda)$  is  $i$ -isomorphic to  $\mathfrak{g}_4^s(1) = \mathfrak{g}_{4,2}^s$ .

**Proposition 4.17.** *Let  $\mathfrak{g} \in \mathcal{D}_{\text{red}}(2+2n)$  then  $\mathfrak{g}$  is an amalgamated product of quadratic Lie superalgebras all  $i$ -isomorphic to  $\mathfrak{g}_{4,2}^s$ .*

Finally, for the invertible case, we recall the matrix  $J_p(\lambda) = \text{diag}_p(\lambda, \dots, \lambda) + J_p$ ,  $p \geq 1$ ,  $\lambda \in \mathbb{C}$  and set

$$\overline{C}_{2p}^J(\lambda) = \begin{pmatrix} J_p(\lambda) & 0 \\ 0 & -{}^t J_p(\lambda) \end{pmatrix}$$

in a Darboux basis of  $\mathbb{C}^{2p}$  then  $\overline{C}_{2p}^J(\lambda) \in \mathfrak{sp}(2p)$ . Let  $\mathfrak{j}_{2p}(\lambda)$  be the double extension of  $\mathbb{C}^{2p}$  by  $\overline{C}_{2p}^J(\lambda)$  then it is called a *Jordan-type quadratic Lie superalgebra*.

When  $\lambda = 0$  and  $p \geq 2$ , we recover the nilpotent Jordan-type Lie superalgebras  $\mathfrak{j}_{2p}$ . If  $\lambda \neq 0$ ,  $\mathfrak{j}_{2p}(\lambda)$  becomes an invertible singular quadratic Lie superalgebra and

$$\mathfrak{j}_{2p}(-\lambda) \simeq \mathfrak{j}_{2p}(\lambda).$$

**Proposition 4.18.** *Let  $\mathfrak{g} \in \mathcal{S}_{\text{inv}}(2+2n)$  then  $\mathfrak{g}$  is an amalgamated product of quadratic Lie superalgebras all  $i$ -isomorphic to Jordan-type quadratic Lie superalgebras  $\mathfrak{j}_{2p}(\lambda)$ , with  $\lambda \neq 0$ .*

#### 4.2. Quadratic dimension of reduced quadratic Lie superalgebras having the 2-dimensional even part.

Let  $(\mathfrak{g}, B)$  be a quadratic Lie superalgebra. To any bilinear form  $B'$  on  $\mathfrak{g}$ , there is an associated map  $D : \mathfrak{g} \rightarrow \mathfrak{g}$  satisfying

$$B'(X, Y) = B(D(X), Y), \quad \forall X, Y \in \mathfrak{g}.$$

**Lemma 4.19.** *If  $B'$  is even then  $D$  is even.*

*Proof.* Let  $X$  be an element in  $\mathfrak{g}_{\bar{0}}$  and assume that  $D(X) = Y + Z$  with  $Y \in \mathfrak{g}_{\bar{0}}$  and  $Z \in \mathfrak{g}_{\bar{1}}$ . Since  $B'$  is even then  $B'(X, \mathfrak{g}_{\bar{1}}) = 0$ . It implies that  $B(D(X), \mathfrak{g}_{\bar{1}}) = B(Z, \mathfrak{g}_{\bar{1}}) = 0$ . By the non-degeneracy of  $B$  on  $\mathfrak{g}_{\bar{1}}$ , we obtain  $Z = 0$  and then  $D(\mathfrak{g}_{\bar{0}}) \subset \mathfrak{g}_{\bar{0}}$ . Similarly to the case  $X \in \mathfrak{g}_{\bar{1}}$ , it concludes that  $D(\mathfrak{g}_{\bar{1}}) \subset \mathfrak{g}_{\bar{1}}$ . Thus,  $D$  is even.  $\square$

**Lemma 4.20.** *Let  $(\mathfrak{g}, B)$  be a quadratic Lie superalgebra,  $B'$  be an even bilinear form on  $\mathfrak{g}$  and  $D \in \mathcal{L}(\mathfrak{g})$  be its associated map. Then:*

- (1)  *$B'$  is invariant if and only if  $D$  satisfies*

$$D([X, Y]) = [D(X), Y] = [X, D(Y)], \quad \forall X, Y \in \mathfrak{g}.$$

- (2)  *$B'$  is supersymmetric if and only if  $D$  satisfies*

$$B(D(X), Y) = B(X, D(Y)), \quad \forall X, Y \in \mathfrak{g}.$$

*In this case,  $D$  is called symmetric.*

- (3)  *$B'$  is non-degenerate if and only if  $D$  is invertible.*

*Proof.* Let  $X, Y$  and  $Z$  be homogeneous elements in  $\mathfrak{g}$  of degrees  $x, y$  and  $z$ , respectively.

- (1) If  $B'$  is invariant then

$$B'([X, Y], Z) = B'(X, [Y, Z]).$$

That means  $B(D([X, Y]), Z) = B(D(X), [Y, Z]) = B([D(X), Y], Z)$ . Since  $B$  is non-degenerate, one has  $D([X, Y]) = [D(X), Y]$ . As a consequence,  $D([X, Y]) = -(-1)^{xy}D([Y, X]) = -(-1)^{xy}[D(Y), X] = [X, D(Y)]$  by  $D$  even.

Conversely, if  $D$  satisfies  $D([X, Y]) = [D(X), Y] = [X, D(Y)]$ , for all  $X, Y \in \mathfrak{g}$ , it is easy to check that  $B'$  is invariant.

- (2)  $B'$  is supersymmetric if and only if  $B'(X, Y) = (-1)^{xy}B'(Y, X)$ . Therefore,  $B(D(X), Y) = (-1)^{xy}B(D(Y), X) = B(X, D(Y))$  by  $B$  supersymmetric.
- (3) It is obvious since  $B$  is non-degenerate.  $\square$

**Definition 4.21.** Let  $\mathfrak{g}$  be a quadratic Lie superalgebra. An even and symmetric map  $D \in \mathcal{L}(\mathfrak{g})$  satisfying Lemma 4.20 (1) is called a *centromorphism* of  $\mathfrak{g}$ .

By [Ben03], given a quadratic Lie superalgebra  $\mathfrak{g}$ , the space of centromorphisms of  $\mathfrak{g}$  and the space generated by invertible ones are the same, denote it by  $\mathcal{C}(\mathfrak{g})$ . As a consequence, the space of even invariant supersymmetric bilinear forms on  $\mathfrak{g}$  coincides with its subspace generated by non-degenerate ones. Moreover, all those spaces have the same dimension called the *quadratic dimension* of  $\mathfrak{g}$  and denoted by



$d_q(\mathfrak{g})$ . The following proposition gives the formula of  $d_q(\mathfrak{g})$  for reduced quadratic Lie superalgebras with 2-dimensional even part.

**Proposition 4.22.** *Let  $\mathfrak{g}$  be a reduced quadratic Lie superalgebra with 2-dimensional even part and  $D \in \mathcal{L}(\mathfrak{g})$  be an even symmetric map. Then:*

- (1)  *$D$  is a centromorphism if and only if there exist  $\mu \in \mathbb{C}$  and an even symmetric map  $Z : \mathfrak{g} \rightarrow \mathcal{Z}(\mathfrak{g})$  such that  $Z|_{[\mathfrak{g}, \mathfrak{g}]} = 0$  and  $D = \mu \text{Id} + Z$ . Moreover  $D$  is invertible if and only if  $\mu \neq 0$ .*

(2)

$$d_q(\mathfrak{g}) = 2 + \frac{(\dim(\mathcal{Z}(\mathfrak{g}) - 1))(\dim(\mathcal{Z}(\mathfrak{g}) - 2))}{2}.$$

*Proof.* The proof goes exactly as Proposition 7.2 given in [DPU], the reader may refer to it. □

## 5. SINGULAR QUADRATIC LIE SUPERALGEBRAS OF TYPE $S_1$

Let  $\mathfrak{g}$  be a singular quadratic Lie superalgebra of type  $S_1$  such that  $\mathfrak{g}_{\bar{0}}$  is non-Abelian. If  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}$  then  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}] = \{0\}$  and therefore  $\mathfrak{g}$  is an orthogonal direct sum of a singular quadratic Lie algebra of type  $S_1$  and a vector space. There is nothing to do. We can assume that  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$ . Fix  $\alpha \in \mathcal{V}_I$  and choose  $\Omega_0 \in \text{Alt}^2(\mathfrak{g}_{\bar{0}})$ ,  $\Omega_1 \in \text{Sym}^2(\mathfrak{g}_{\bar{1}})$  such that

$$I = \alpha \wedge \Omega_0 + \alpha \otimes \Omega_1.$$

Let  $X_{\bar{0}} = \phi^{-1}(\alpha)$  then  $X_{\bar{0}} \in \mathcal{Z}(\mathfrak{g})$  and  $B(X_{\bar{0}}, X_{\bar{0}}) = 0$ . We define linear maps  $C_0 : \mathfrak{g}_{\bar{0}} \rightarrow \mathfrak{g}_{\bar{0}}$ ,  $C_1 : \mathfrak{g}_{\bar{1}} \rightarrow \mathfrak{g}_{\bar{1}}$  by  $\Omega_0(X, Y) = B(C_0(X), Y)$  if  $X, Y \in \mathfrak{g}_{\bar{0}}$  and  $\Omega_1(X, Y) = B(C_1(X), Y)$  if  $X, Y \in \mathfrak{g}_{\bar{1}}$ . Let  $C : \mathfrak{g} \rightarrow \mathfrak{g}$  defined by  $C(X + Y) = C_0(X) + C_1(Y)$ , for all  $X \in \mathfrak{g}_{\bar{0}}$ ,  $Y \in \mathfrak{g}_{\bar{1}}$ .

**Proposition 5.1.** *For all  $X, Y \in \mathfrak{g}$ , the Lie super-bracket of  $\mathfrak{g}$  is defined by:*

$$[X, Y] = B(X_{\bar{0}}, X)C(Y) - B(X_{\bar{0}}, Y)C(X) + B(C(X), Y)X_{\bar{0}}.$$

*In particular, if  $X, Y \in \mathfrak{g}_{\bar{0}}$ ,  $Z, T \in \mathfrak{g}_{\bar{1}}$  then*

- (1)  $[X, Y] = B(X_{\bar{0}}, X)C_0(Y) - B(X_{\bar{0}}, Y)C_0(X) + B(C_0(X), Y)X_{\bar{0}}$ ,
- (2)  $[X, Z] = B(X_{\bar{0}}, X)C_1(Z)$ ,
- (3)  $[Z, T] = B(C_1(Z), T)X_{\bar{0}}$

*Proof.* By Proposition 2.7,  $\mathfrak{g}_{\bar{0}}$  is a singular quadratic Lie algebra so the assertion (1) follows [DPU]. Given  $X \in \mathfrak{g}_{\bar{0}}$ ,  $Y, Z \in \mathfrak{g}_{\bar{1}}$ , one has

$$B([X, Y], Z) = \alpha \otimes \Omega_1(X, Y, Z) = \alpha(X)\Omega_1(Y, Z) = B(X_{\bar{0}}, X)B(C_1(Y), Z).$$

Hence we obtain (2) and (3). □

Now, we show that  $\mathfrak{g}_{\bar{0}}$  is solvable. Consider the quadratic Lie algebra  $\mathfrak{g}_{\bar{0}}$  with 3-form  $I_0 = \alpha \wedge \Omega_0$ . Write  $\Omega_0 = \sum_{i < j} a_{ij} \alpha_i \wedge \alpha_j$ , with  $a_{ij} \in \mathbb{C}$ . Set  $X_i = \phi^{-1}(\alpha_i)$  then

$$C_0 = \sum_{i < j} a_{ij} (\alpha_i \otimes X_j - \alpha_j \otimes X_i).$$

Define the space  $W_{I_0} \subset \mathfrak{g}_0^*$  by:

$$\mathcal{W}_{I_0} = \{\iota_{X \wedge Y}(I_0) \mid X, Y \in \mathfrak{g}_0\}.$$

Then  $\mathcal{W}_{I_0} = \phi([\mathfrak{g}_0, \mathfrak{g}_0])$  and that implies  $\text{Im}(C_0) \subset [\mathfrak{g}_0, \mathfrak{g}_0]$ . In Section 2, it is known that  $\{\alpha, I_0\} = 0$  and then  $[X_0, \mathfrak{g}_0] = 0$ . As a sequence,  $B(X_0, [\mathfrak{g}_0, \mathfrak{g}_0]) = 0$ . That deduces  $B(X_0, \text{Im}(C_0)) = 0$ . Therefore  $[[\mathfrak{g}_0, \mathfrak{g}_0], [\mathfrak{g}_0, \mathfrak{g}_0]] = [\text{Im}(C_0), \text{Im}(C_0)] \subset \mathbb{C}X_0 \subset \mathcal{Z}(\mathfrak{g})$  and we conclude that  $\mathfrak{g}_0$  is solvable.

By  $B$  non-degenerate there is an element  $Y_0 \in \mathfrak{g}_0$  isotropic such that  $B(X_0, Y_0) = 1$ . Moreover, combined with  $\mathfrak{g}_0$  a solvable singular quadratic Lie algebra, we can choose  $Y_0$  satisfying  $C_0(Y_0) = 0$  and we obtain then a straightforward consequence as follows:

**Corollary 5.2.**

- (1)  $C = \text{ad}(Y_0)$ ,  $\ker(C) = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_0$  and  $[\mathfrak{g}, \mathfrak{g}] = \text{Im}(C) \oplus \mathbb{C}X_0$ .
- (2) The Lie superalgebra  $\mathfrak{g}$  is solvable. Moreover,  $\mathfrak{g}$  is nilpotent if and only if  $C$  is nilpotent.

**5.1. Singular quadratic Lie superalgebras of type  $S_1$  and double extensions.**

The description of the Lie super-bracket in Proposition 5.1 allows us to propose a definition of double extension of a quadratic  $\mathbb{Z}_2$ -graded vector space as follows:

**Definition 5.3.** Let  $(\mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}_1, B_{\mathfrak{q}})$  be a quadratic  $\mathbb{Z}_2$ -graded vector space and  $\overline{C}$  be an even endomorphism of  $\mathfrak{q}$ . Assume that  $\overline{C}$  is skew-supersymmetric, that is,  $B(\overline{C}(X), Y) = -B(X, \overline{C}(Y))$ , for all  $X, Y \in \mathfrak{q}$ . Let  $(\mathfrak{t} = \text{span}\{X_0, Y_0\}, B_{\mathfrak{t}})$  be a 2-dimensional quadratic vector space with the symmetric bilinear form  $B_{\mathfrak{t}}$  defined by:

$$B_{\mathfrak{t}}(X_0, X_0) = B_{\mathfrak{t}}(Y_0, Y_0) = 0 \text{ and } B_{\mathfrak{t}}(X_0, Y_0) = 1.$$

Consider the vector space  $\mathfrak{g} = \mathfrak{t} \oplus^{\perp} \mathfrak{q}$  equipped with the bilinear form  $B = B_{\mathfrak{t}} + B_{\mathfrak{q}}$  and define on  $\mathfrak{g}$  the following bracket:

$$[\lambda X_0 + \mu Y_0 + X, \lambda' X_0 + \mu' Y_0 + Y] = \mu \overline{C}(Y) - \mu' \overline{C}(X) + B(\overline{C}(X), Y) X_0,$$

for all  $X, Y \in \mathfrak{q}, \lambda, \mu, \lambda', \mu' \in \mathbb{C}$ . Then  $(\mathfrak{g}, B)$  is a quadratic solvable Lie superalgebra with  $\mathfrak{g}_0 = \mathfrak{t} \oplus \mathfrak{q}_0$  and  $\mathfrak{g}_1 = \mathfrak{q}_1$ . We say that  $\mathfrak{g}$  is the *double extension* of  $\mathfrak{q}$  by  $\overline{C}$ .

Note that an even skew-supersymmetric endomorphism  $\overline{C}$  on  $\mathfrak{q}$  can be written by  $\overline{C} = \overline{C}_0 + \overline{C}_1$  where  $\overline{C}_0 \in \mathfrak{o}(\mathfrak{q}_0)$  and  $\overline{C}_1 \in \mathfrak{sp}(\mathfrak{q}_1)$ .

**Corollary 5.4.** Let  $\mathfrak{g}$  be the double extension of  $\mathfrak{q}$  by  $\overline{C}$ . Denote by  $C = \text{ad}(Y_0)$  then one has

- (1)  $[X, Y] = B(X_0, X)C(Y) - B(X_0, Y)C(X) + B(C(X), Y)X_0$ , for all  $X, Y \in \mathfrak{g}$ .
- (2)  $\mathfrak{g}$  is a singular quadratic Lie superalgebra. If  $\overline{C}|_{\mathfrak{q}_1}$  is nonzero then  $\mathfrak{g}$  is of type  $S_1$ .

*Proof.* The assertion (1) is direct from the above definition. Let  $\alpha = \phi(X_0)$  and define the bilinear form  $\Omega : \mathfrak{g} \rightarrow \mathfrak{g}$  by  $\Omega(X, Y) = B(C(X), Y)$  for all  $X, Y \in \mathfrak{g}$ . By  $B$  even and supersymmetric,  $C$  even and skew-supersymmetric (with respect to  $B$ )

then  $\Omega = \Omega_0 + \Omega_1 \in \text{Alt}^2(\mathfrak{g}_{\bar{0}}) \oplus \text{Sym}^2(\mathfrak{g}_{\bar{1}})$ . The formula in (1) can be replaced by  $I = \alpha \wedge \Omega_0 + \alpha \otimes \Omega_1 = \alpha \wedge \Omega$ . Therefore,  $\text{dup}(\mathfrak{g}) \geq 1$  and  $\mathfrak{g}$  is singular. If  $\bar{C}|_{\mathfrak{q}_{\bar{1}}}$  is nonzero then  $\Omega_1 \neq 0$ . In this case,  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$  and thus  $\text{dup}(\mathfrak{g}) = 1$ .  $\square$

As a consequence of Proposition 5.1 and Definition 5.3, one has

**Lemma 5.5.** *Let  $(\mathfrak{g}, B)$  be a singular quadratic Lie superalgebra of type  $S_1$ . Keep the notations as in Proposition 5.1 and Corollary 5.2. Then  $(\mathfrak{g}, B)$  is the double extension of  $\mathfrak{q} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}})^{\perp}$  by  $\bar{C} = C|_{\mathfrak{q}}$ .*

*Remark 5.6.* The above definition is a generalization of the definition of double extension of a quadratic vector space by a skew-symmetric map in [DPU] and Definition 4.6. Moreover, if let  $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}})^{\perp} \oplus (\mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}})$  be the double extension of  $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$  by  $\bar{C} = \bar{C}_0 + \bar{C}_1$  then  $\mathfrak{g}_{\bar{0}}$  is the double extension of  $\mathfrak{q}_{\bar{0}}$  by  $\bar{C}_0$  and the subalgebra  $(\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}})^{\perp} \oplus \mathfrak{q}_{\bar{1}}$  is the double extension of  $\mathfrak{q}_{\bar{1}}$  by  $\bar{C}_1$ .

The proof of the proposition below is completely analogous to the proof of Proposition 4.10, so we omit it.

**Proposition 5.7.** *Let  $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}})^{\perp} \oplus (\mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}})$  and  $\mathfrak{g}' = (\mathbb{C}X'_{\bar{0}} \oplus \mathbb{C}Y'_{\bar{0}})^{\perp} \oplus (\mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}})$  be two double extensions of  $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$  by  $\bar{C} = \bar{C}_0 + \bar{C}_1$  and  $\bar{C}' = \bar{C}'_0 + \bar{C}'_1$ , respectively. Assume that  $\bar{C}_1$  is nonzero. Then*

- (1) *there exists a Lie superalgebra isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}'$  if and only if there exist invertible maps  $P \in \mathcal{L}(\mathfrak{q}_{\bar{0}})$ ,  $Q \in \mathcal{L}(\mathfrak{q}_{\bar{1}})$  and a nonzero  $\lambda \in \mathbb{C}$  such that*
  - (i)  $\bar{C}'_0 = \lambda P \bar{C}_0 P^{-1}$  and  $P^* P \bar{C}_0 = \bar{C}_0$ .
  - (ii)  $\bar{C}'_1 = \lambda Q \bar{C}_1 Q^{-1}$  and  $Q^* Q \bar{C}_1 = \bar{C}_1$ .*where  $P^*$  and  $Q^*$  are the adjoint maps of  $P$  and  $Q$  with respect to  $B|_{\mathfrak{q}_{\bar{0}} \times \mathfrak{q}_{\bar{0}}}$  and  $B|_{\mathfrak{q}_{\bar{1}} \times \mathfrak{q}_{\bar{1}}}$ .*
- (2) *there exists an i-isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}'$  if and only if there is a nonzero  $\lambda \in \mathbb{C}$  such that  $\bar{C}'_0$  is in the  $O(\mathfrak{q}_{\bar{0}})$ -adjoint orbit through  $\lambda \bar{C}_0$  and  $\bar{C}'_1$  is in the  $\text{Sp}(\mathfrak{q}_{\bar{1}})$ -adjoint orbit through  $\lambda \bar{C}_1$ .*

*Remark 5.8.* If let  $M = P + Q$  then  $M^{-1} = P^{-1} + Q^{-1}$  and  $M^* = P^* + Q^*$ . The formulas in Proposition 5.7 (1) can be written:

$$\bar{C}' = \lambda M \bar{C} M^{-1} \text{ and } M^* M \bar{C} = \bar{C}.$$

Hence, the classification problem of singular quadratic Lie superalgebras of type  $S_1$  (up to i-isomorphism) can be reduced to the classification of  $O(\mathfrak{q}_{\bar{0}}) \times \text{Sp}(\mathfrak{q}_{\bar{1}})$ -orbits of  $\mathfrak{o}(\mathfrak{q}_{\bar{0}}) \oplus \mathfrak{sp}(\mathfrak{q}_{\bar{1}})$ , where  $O(\mathfrak{q}_{\bar{0}}) \times \text{Sp}(\mathfrak{q}_{\bar{1}})$  denotes the direct product of two groups  $O(\mathfrak{q}_{\bar{0}})$  and  $\text{Sp}(\mathfrak{q}_{\bar{1}})$ .

**Definition 5.9.** Let  $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$  be a quadratic  $\mathbb{Z}_2$ -graded vector space. An even isomorphism  $F \in \mathcal{L}(\mathfrak{q})$  is called an *isometry* of  $\mathfrak{q}$  if  $F|_{\mathfrak{q}_{\bar{0}}}$  and  $F|_{\mathfrak{q}_{\bar{1}}}$  are isometries.

To prove the following Corollary, it is enough to follow exactly the same steps as in Corollary 4.11.

**Corollary 5.10.** *Let  $(\mathfrak{g}, B)$  and  $(\mathfrak{g}', B')$  be double extensions of  $(\mathfrak{q}, \overline{B})$  and  $(\mathfrak{q}', \overline{B}')$  by  $\overline{C}$  and  $\overline{C}'$  respectively where  $\overline{B} = B|_{\mathfrak{q} \times \mathfrak{q}}$  and  $\overline{B}' = B'|_{\mathfrak{q}' \times \mathfrak{q}'}$ . Write  $\mathfrak{g} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$  and  $\mathfrak{g}' = (\mathbb{C}X'_{\overline{0}} \oplus \mathbb{C}Y'_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}'$ . Then:*

- (1) *there exists an i-isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}'$  if and only if there exists an isometry  $\overline{A} : \mathfrak{q} \rightarrow \mathfrak{q}'$  such that  $\overline{C}' = \lambda \overline{A} \overline{C} \overline{A}^{-1}$ , for some nonzero  $\lambda \in \mathbb{C}$ .*
- (2) *there exists a Lie superalgebra isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}'$  if and only if there exist even invertible maps  $\overline{Q} : \mathfrak{q} \rightarrow \mathfrak{q}'$  and  $\overline{P} \in \mathcal{L}(\mathfrak{q})$  such that*
  - (i)  $\overline{C}' = \lambda \overline{Q} \overline{C} \overline{Q}^{-1}$  for some nonzero  $\lambda \in \mathbb{C}$ ,
  - (ii)  $\overline{P}^* \overline{P} \overline{C} = \overline{C}$  and
  - (iii)  $\overline{Q} \overline{P}^{-1}$  is an isometry from  $\mathfrak{q}$  onto  $\mathfrak{q}'$ .

## 5.2. Fitting decomposition of a skew-supersymmetric map.

We recall the following useful result (see for instance [DPU]):

**Lemma 5.11.** *Let  $\overline{C}$  and  $\overline{C}'$  be nilpotent elements in  $\mathfrak{o}(n)$ . Then  $\overline{C}$  is conjugate to  $\lambda \overline{C}'$  modulo  $O(n)$  for some nonzero  $\lambda \in \mathbb{C}$  if and only if  $\overline{C}$  is conjugate to  $\overline{C}'$ .*

Remark that the lemma remains valid if we replace  $\mathfrak{o}(n)$  by  $\mathfrak{sp}(2n)$  and  $O(n)$  by  $Sp(2n)$ .

**Proposition 5.12.** *Let  $\mathfrak{g}$  and  $\mathfrak{g}'$  be two nilpotent singular quadratic Lie superalgebras. Then  $\mathfrak{g}$  and  $\mathfrak{g}'$  are isomorphic if and only if they are i-isomorphic.*

*Proof.* Singular quadratic Lie superalgebras  $\mathfrak{g}$  and  $\mathfrak{g}'$  are regarded as double extensions  $\mathfrak{g} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$  and  $(\mathbb{C}X'_{\overline{0}} \oplus \mathbb{C}Y'_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}'$  by  $\overline{C}$  and  $\overline{C}'$  where  $\mathfrak{q} = \mathfrak{q}_{\overline{0}} \oplus \mathfrak{q}_{\overline{1}}$  and  $\mathfrak{q}' = \mathfrak{q}'_{\overline{0}} \oplus \mathfrak{q}'_{\overline{1}}$ . By Corollary 5.2,  $\overline{C}$  and  $\overline{C}'$  are nilpotent. Rewrite  $\overline{C} = \overline{C}_0 + \overline{C}_1$  and  $\overline{C}' = \overline{C}'_0 + \overline{C}'_1$ , where  $\overline{C}_0 \in \mathfrak{o}(\mathfrak{q}_{\overline{0}})$ ,  $\overline{C}'_0 \in \mathfrak{o}(\mathfrak{q}'_{\overline{0}})$ ,  $\overline{C}_1 \in \mathfrak{sp}(\mathfrak{q})$  and  $\overline{C}'_1 \in \mathfrak{sp}(\mathfrak{q}')$ .

If  $\mathfrak{g}$  and  $\mathfrak{g}'$  are isomorphic then  $\dim(\mathfrak{q}_{\overline{0}}) = \dim(\mathfrak{q}'_{\overline{0}})$  and  $\dim(\mathfrak{q}_{\overline{1}}) = \dim(\mathfrak{q}'_{\overline{1}})$ . Thus, there exist isometries  $\overline{F}_0 : \mathfrak{q}'_{\overline{0}} \rightarrow \mathfrak{q}_{\overline{0}}$  and  $\overline{F}_1 : \mathfrak{q}'_{\overline{1}} \rightarrow \mathfrak{q}_{\overline{1}}$  and then we define an isometry  $\overline{F} : \mathfrak{q}' \rightarrow \mathfrak{q}$  by  $\overline{F}(X' + Y') = \overline{F}_0(X') + \overline{F}_1(Y')$  for all  $X' \in \mathfrak{q}'_{\overline{0}}$  and  $Y' \in \mathfrak{q}'_{\overline{1}}$ . We now set  $F : \mathfrak{g}' \rightarrow \mathfrak{g}$  by  $F(X'_{\overline{0}}) = X_{\overline{0}}$ ,  $F(Y'_{\overline{0}}) = Y_{\overline{0}}$ ,  $F|_{\mathfrak{q}'} = \overline{F}$  and a new Lie super-bracket on  $\mathfrak{g}$  by:

$$[X, Y]'' = F([F^{-1}(X), F^{-1}(Y)]'), \quad \forall X, Y \in \mathfrak{g}.$$

Denote by  $\mathfrak{g}''$  this new quadratic Lie superalgebras. It is easy to see that  $\mathfrak{g}'' = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \overset{\perp}{\oplus} \mathfrak{q}$  is the double extension of  $\mathfrak{q}$  by  $\overline{C}'' = \overline{F} \overline{C}' \overline{F}^{-1}$  and  $\mathfrak{g}''$  is i-isomorphic to  $\mathfrak{g}'$ . It need to prove that  $\mathfrak{g}''$  is i-isomorphic to  $\mathfrak{g}$ . Write  $\overline{C}'' = \overline{C}''_0 + \overline{C}''_1 \in \mathfrak{o}(\mathfrak{q}_{\overline{0}}) \oplus \mathfrak{sp}(\mathfrak{q}_{\overline{1}})$ . Since  $\mathfrak{g}$  and  $\mathfrak{g}''$  are isomorphic then there exist invertible maps  $P : \mathfrak{q}_{\overline{0}} \rightarrow \mathfrak{q}_{\overline{0}}$  and  $Q : \mathfrak{q}_{\overline{1}} \rightarrow \mathfrak{q}_{\overline{1}}$  such that  $\overline{C}''_0 = \lambda \overline{P} \overline{C}_0 \overline{P}^{-1}$  and  $\overline{C}''_1 = \lambda \overline{Q} \overline{C}_1 \overline{Q}^{-1}$  for some nonzero  $\lambda \in \mathbb{C}$ . By Lemma 5.11,  $\overline{C}_0$  and  $\overline{C}''_0$  are conjugate under  $O(\mathfrak{q}_{\overline{0}})$ ,  $\overline{C}_1$  and  $\overline{C}''_1$  are conjugate under  $Sp(\mathfrak{q}_{\overline{1}})$  and we can assume that  $\lambda = 1$ . Therefore  $\mathfrak{g}$  and  $\mathfrak{g}''$  are i-isomorphic. The proposition is proved.  $\square$

Let now  $\mathfrak{g}$  be a singular quadratic Lie superalgebra of type  $S_1$ . Write  $\mathfrak{g}$  as a double extension of  $(\mathfrak{q} = \mathfrak{q}_{\overline{0}} \oplus \mathfrak{q}_{\overline{1}}, \overline{B})$  by  $\overline{C} = \overline{C}_0 + \overline{C}_1$  where  $\overline{C} = \text{ad}(Y_{\overline{0}})|_{\mathfrak{q}}$ ,  $\overline{C}_0 = \overline{C}|_{\mathfrak{q}_{\overline{0}}}$

and  $\overline{C}_1 = \overline{C}|_{\mathfrak{q}_1}$ . We consider the Fitting decomposition of  $\overline{C}_0$  on  $\mathfrak{q}_0$  and  $\overline{C}_1$  on  $\mathfrak{q}_1$  by:

$$\mathfrak{q}_0 = \mathfrak{q}_0^N \oplus \mathfrak{q}_0^I \quad \text{and} \quad \mathfrak{q}_1 = \mathfrak{q}_1^N \oplus \mathfrak{q}_1^I$$

where  $\mathfrak{q}_0^N$  and  $\mathfrak{q}_0^I$  (resp.  $\mathfrak{q}_1^N$  and  $\mathfrak{q}_1^I$ ) are  $\overline{C}_0$ -stable (resp.  $\overline{C}_1$ -stable),  $\overline{C}_0^N = \overline{C}_0|_{\mathfrak{q}_0^N}$  and  $\overline{C}_1^N = \overline{C}_1|_{\mathfrak{q}_1^N}$  are nilpotent,  $\overline{C}_0^I = \overline{C}_0|_{\mathfrak{q}_0^I}$  and  $\overline{C}_1^I = \overline{C}_1|_{\mathfrak{q}_1^I}$  are invertible. Recall that  $\overline{C}$  is skew-supersymmetric then  $\mathfrak{q}_0^I = (\mathfrak{q}_0^N)^\perp$  in  $\mathfrak{g}_0$  and  $\mathfrak{q}_1^I = (\mathfrak{q}_1^N)^\perp$  in  $\mathfrak{g}_1$ .

Next, we set

$$\mathfrak{q}_N = \mathfrak{q}_0^N \oplus \mathfrak{q}_1^N \quad \text{and} \quad \mathfrak{q}_I = \mathfrak{q}_0^I \oplus \mathfrak{q}_1^I$$

As a consequence,  $\overline{C}_N = \overline{C}|_{\mathfrak{q}_N}$  is nilpotent,  $\overline{C}_I = \overline{C}|_{\mathfrak{q}_I}$  is invertible,  $[\mathfrak{q}_N, \mathfrak{q}_I] = \{0\}$ , the restrictions  $\overline{B}_N = \overline{B}|_{\mathfrak{q}_N \times \mathfrak{q}_N}$  and  $\overline{B}_I = \overline{B}|_{\mathfrak{q}_I \times \mathfrak{q}_I}$  are non-degenerate and supersymmetric. It is easy to check that  $\overline{C}_N = \overline{C}_0^N + \overline{C}_1^N$ ,  $\overline{C}_I = \overline{C}_0^I + \overline{C}_1^I$ . Moreover,  $\overline{C}_N$ ,  $\overline{C}_I$  are skew-supersymmetric and they are Fitting components of  $\overline{C}$  in  $\mathfrak{q}$ . Let  $\mathfrak{g}_N = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \oplus^\perp \mathfrak{q}_N$  and  $\mathfrak{g}_I = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \oplus^\perp \mathfrak{q}_I$ . Then  $\mathfrak{g}_N$  and  $\mathfrak{g}_I$  are subalgebras of  $\mathfrak{g}$ ,  $\mathfrak{g}_N$  is the double extension of  $\mathfrak{q}_N$  by  $\overline{C}_N$ ,  $\mathfrak{g}_I$  is the double extension of  $\mathfrak{q}_I$  by  $\overline{C}_I$  and  $\mathfrak{g}_N$  is a nilpotent singular quadratic Lie superalgebra.

**Definition 5.13.** The subalgebras  $\mathfrak{g}_N$  and  $\mathfrak{g}_I$  as above are respectively the *nilpotent* and *invertible Fitting components* of  $\mathfrak{g}$ .

**Definition 5.14.** A double extension is called an *invertible quadratic Lie superalgebra* if the corresponding skew-supersymmetric map is invertible.

It is easy to check that the dimension of an invertible quadratic Lie superalgebra must be even. Moreover, following Corollary 5.10, two invertible quadratic Lie superalgebras are isomorphic if and only if they are i-isomorphic. This property is still right for singular quadratic Lie superalgebras of type  $S_1$ .

**Proposition 5.15.** Let  $\mathfrak{g}$  and  $\mathfrak{g}'$  be singular quadratic Lie superalgebras of type  $S_1$  and  $\mathfrak{g}_N$ ,  $\mathfrak{g}_I$ ,  $\mathfrak{g}'_N$ ,  $\mathfrak{g}'_I$  be their Fitting components, respectively. Then

- (1)  $\mathfrak{g} \stackrel{i}{\simeq} \mathfrak{g}'$  if and only if  $\mathfrak{g}_N \stackrel{i}{\simeq} \mathfrak{g}'_N$  and  $\mathfrak{g}_I \stackrel{i}{\simeq} \mathfrak{g}'_I$ . The result remains valid if we replace  $\stackrel{i}{\simeq}$  by  $\simeq$ .
- (2)  $\mathfrak{g}$  and  $\mathfrak{g}'$  are isomorphic if and only if they are i-isomorphic.

*Proof.* The proposition is proved as Proposition 6.4 in [DPU]. It is sketched as follows. We assume that  $\mathfrak{g} \simeq \mathfrak{g}'$ . They are regarded as double extensions  $\mathfrak{g} = (\mathbb{C}X_0 \oplus \mathbb{C}Y_0) \oplus^\perp \mathfrak{q}$  and  $(\mathbb{C}X'_0 \oplus \mathbb{C}Y'_0) \oplus^\perp \mathfrak{q}'$  by  $\overline{C}$  and  $\overline{C}'$ . By Corollary 5.10, there are an even invertible map  $\overline{P} : \mathfrak{q} \rightarrow \mathfrak{q}'$  and a nonzero  $\lambda \in \mathbb{C}$  such that  $\overline{C}' = \lambda \overline{P} \overline{C} \overline{P}^{-1}$ , so  $\mathfrak{q}'_N = \overline{P}(\mathfrak{q}_N)$  and  $\mathfrak{q}'_I = \overline{P}(\mathfrak{q}_I)$ , then  $\dim(\mathfrak{q}'_N) = \dim(\mathfrak{q}_N)$  and  $\dim(\mathfrak{q}'_I) = \dim(\mathfrak{q}_I)$ . Thus, there exist isometries  $F_N : \mathfrak{q}'_N \rightarrow \mathfrak{q}_N$  and  $F_I : \mathfrak{q}'_I \rightarrow \mathfrak{q}_I$  and we can define an isometry  $\overline{F} : \mathfrak{q}' \rightarrow \mathfrak{q}$  by  $\overline{F}(X'_N + X'_I) = F_N(X'_N) + F_I(X'_I)$ ,  $\forall X'_N \in \mathfrak{q}'_N$  and  $X'_I \in \mathfrak{q}'_I$ . We now define  $F : \mathfrak{g}' \rightarrow \mathfrak{g}$  by  $F(X'_1) = X_1$ ,  $F(Y'_1) = Y_1$ ,  $F|_{\mathfrak{q}'} = \overline{F}$  and a new Lie super-bracket on  $\mathfrak{g}$ :

$$[X, Y]'' = F([F^{-1}(X), F^{-1}(Y)]'), \quad \forall X, Y \in \mathfrak{g}.$$

Denote by  $\mathfrak{g}''$  this new quadratic Lie superalgebra. It is obvious that  $\mathfrak{g}' \stackrel{i}{\simeq} \mathfrak{g}''$ . It remains to prove the assertions for two quadratic Lie superalgebras  $\mathfrak{g}$  and  $\mathfrak{g}''$ . Those follow Corollary 5.10, Lemma 5.11 and Proposition 5.12.  $\square$

**Proposition 5.16.** *The dup-number is invariant under Lie superalgebra isomorphisms, i.e. if  $(\mathfrak{g}, B)$  and  $(\mathfrak{g}', B')$  are quadratic Lie superalgebras with  $\mathfrak{g} \simeq \mathfrak{g}'$ , then  $\text{dup}(\mathfrak{g}) = \text{dup}(\mathfrak{g}')$ .*

*Proof.* By Lemma 2.4 we can assume that  $\mathfrak{g}$  is reduced. By Proposition 2.3,  $\mathfrak{g}'$  is also reduced. Since  $\mathfrak{g} \simeq \mathfrak{g}'$  then we can identify  $\mathfrak{g} = \mathfrak{g}'$  as a Lie superalgebra equipped with the bilinear forms  $B, B'$  and we have two dup-numbers:  $\text{dup}_B(\mathfrak{g})$  and  $\text{dup}_{B'}(\mathfrak{g})$ .

We start with the case  $\text{dup}_B(\mathfrak{g}) = 3$ . Since  $\mathfrak{g}$  is reduced then  $\mathfrak{g}_{\bar{1}} = \{0\}$  and  $\mathfrak{g}$  is a reduced singular quadratic Lie algebra of type  $S_3$ . By [PU07],  $\dim([\mathfrak{g}, \mathfrak{g}]) = 3$  and then  $\text{dup}_{B'}(\mathfrak{g}) = 3$ .

If  $\text{dup}_B(\mathfrak{g}) = 1$ , then  $\mathfrak{g}$  is of type  $S_1$  with respect to  $B$ . There are two cases:  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}$  and  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$ . If  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}$  then  $\mathfrak{g}_{\bar{1}} = \{0\}$  by  $\mathfrak{g}$  reduced. In this case,  $\mathfrak{g}$  is a reduced singular quadratic Lie algebra of type  $S_1$ . By [DPU],  $\mathfrak{g}$  is also a reduced singular quadratic Lie algebra of type  $S_1$  with the bilinear form  $B'$ , i.e.  $\text{dup}_{B'}(\mathfrak{g}) = 1$ .

Assume that  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq \{0\}$ , we need the following lemma:

**Lemma 5.17.** *Let  $\mathfrak{g}$  be a reduced quadratic Lie superalgebras of type  $S_1$  such that  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq 0$  and  $D \in \mathcal{L}(\mathfrak{g})$  be an even symmetric map. Then  $D$  is a centromorphism if and only if there exist  $\mu \in \mathbb{C}$  and an even symmetric map  $Z : \mathfrak{g} \rightarrow \mathcal{Z}(\mathfrak{g})$  such that  $Z|_{[\mathfrak{g}, \mathfrak{g}]} = 0$  and  $D = \mu \text{Id} + Z$ . Moreover  $D$  is invertible if and only if  $\mu \neq 0$ .*

*Proof.* First,  $\mathfrak{g}$  can be realized as the double extension  $\mathfrak{g} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \oplus^{\perp} \mathfrak{q}$  by  $C = \text{ad}(Y_{\bar{0}})$  and let  $\bar{C} = C|_{\mathfrak{q}}$ .

Assume that  $D$  is an invertible centromorphism. The condition (1) of Lemma 4.20 implies that  $D \circ \text{ad}(X) = \text{ad}(X) \circ D$ , for all  $X \in \mathfrak{g}$  and then  $DC = CD$ . Using formula (1) of Corollary 5.4 and  $CD = DC$ , from  $[D(X), Y_{\bar{0}}] = [X, D(Y_{\bar{0}})]$  we find

$$D(C(X)) = \mu C(X), \forall X \in \mathfrak{g}, \text{ where } \mu = B(D(X_{\bar{0}}), Y_{\bar{0}}).$$

Since  $D$  is invertible, one has  $\mu \neq 0$  and  $C(D - \mu \text{Id}) = 0$ . Recall that  $\ker(C) = \mathbb{C}X_{\bar{0}} \oplus \ker(\bar{C}) \oplus \mathbb{C}Y_{\bar{0}} = \mathcal{Z}(\mathfrak{g}) \oplus \mathbb{C}Y_{\bar{0}}$ , there exist a map  $Z : \mathfrak{g} \rightarrow \mathcal{Z}(\mathfrak{g})$  and  $\varphi \in \mathfrak{g}^*$  such that

$$D - \mu \text{Id} = Z + \varphi \otimes Y_{\bar{0}}.$$

It needs to show that  $\varphi = 0$ . Indeed,  $D$  maps  $[\mathfrak{g}, \mathfrak{g}]$  into itself and  $Y_{\bar{0}} \notin [\mathfrak{g}, \mathfrak{g}]$ , so  $\varphi|_{[\mathfrak{g}, \mathfrak{g}]} = 0$ . One has  $[\mathfrak{g}, \mathfrak{g}] = \mathbb{C}X_{\bar{0}} \oplus \text{Im}(\bar{C})$ . If  $X \in \text{Im}(\bar{C})$ , let  $X = C(Y)$ . Then  $D(X) = D(C(Y)) = \mu C(Y)$ , so  $D(X) = \mu X$ . For  $Y_{\bar{0}}$ ,  $D([Y_{\bar{0}}, X]) = DC(X) = \mu C(X)$  for all  $X \in \mathfrak{g}$ . But also,  $D([Y_{\bar{0}}, X]) = [D(Y_{\bar{0}}), X] = \mu C(X) + \varphi(Y_{\bar{0}})C(X)$ , hence  $\varphi(Y_{\bar{0}}) = 0$ . As a consequence,  $D(Y_{\bar{0}}) = \mu Y_{\bar{0}} + Z(Y_{\bar{0}})$ .

Now, we prove that  $D(X_{\bar{0}}) = \mu X_{\bar{0}}$ . Indeed, since  $D$  is even and  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathbb{C}X_{\bar{0}}$  then one has

$$D(X_{\bar{0}}) \subset D([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]) = [D(\mathfrak{g}_{\bar{1}}), \mathfrak{g}_{\bar{1}}] \subset [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathbb{C}X_{\bar{0}}.$$

It implies that,  $D(X_{\bar{0}}) = aX_{\bar{0}}$ . Combined with  $B(D(Y_{\bar{0}}), X_{\bar{0}}) = B(Y_{\bar{0}}, D(X_{\bar{0}}))$ , we obtain  $\mu = a$ .

Let  $X \in \mathfrak{q}$ ,  $B(D(X_{\bar{0}}), X) = \mu B(X_{\bar{0}}, X) = 0$ . Moreover,  $B(D(X_{\bar{0}}), X) = B(X_{\bar{0}}, D(X))$ , so  $\varphi(X) = 0$ .

Since  $\mathcal{C}(\mathfrak{g})$  is generated by invertible centromorphisms then the necessary condition of Lemma is finished. The sufficiency is obvious.  $\square$

Let us return now to the proposition. By the previous lemma, the bilinear form  $B'$  defines an associated invertible centromorphism  $D = \mu \text{Id} + Z$  for some nonzero  $\mu \in \mathbb{C}$  and  $Z : \mathfrak{g} \rightarrow \mathcal{Z}(\mathfrak{g})$  satisfying  $Z|_{[\mathfrak{g}, \mathfrak{g}]} = 0$ . For all  $X, Y, Z \in \mathfrak{g}$ , one has:

$$I'(X, Y, Z) = B'([X, Y], Z) = B(D([X, Y]), Z) = B([D(X), Y], Z) = \mu B([X, Y], Z).$$

That means  $I' = \mu I$  and then  $\text{dup}_{B'}(\mathfrak{g}) = \text{dup}_B(\mathfrak{g}) = 1$ .

Finally, if  $\text{dup}_B(\mathfrak{g}) = 0$  then  $\mathfrak{g}$  cannot be of type  $S_3$  or  $S_1$  with respect to  $B'$ , so  $\text{dup}_{B'}(\mathfrak{g}) = 0$ .  $\square$

Let  $\mathfrak{g}$  be a reduced singular quadratic Lie superalgebra of type  $S_1$  such that  $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq 0$ . Keep the notation as in Lemma 5.17. We set  $\mathcal{Z}(\mathfrak{g})_{\bar{0}} = \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}}$ ,  $\mathcal{Z}(\mathfrak{g})_{\bar{1}} = \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}}$ ,  $[\mathfrak{g}, \mathfrak{g}]_{\bar{0}} = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_{\bar{0}}$  and  $[\mathfrak{g}, \mathfrak{g}]_{\bar{1}} = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_{\bar{1}}$ . It is obvious that  $X_{\bar{0}} \in \mathcal{Z}(\mathfrak{g})_{\bar{0}} \subset [\mathfrak{g}, \mathfrak{g}]_{\bar{0}}$  and  $\mathcal{Z}(\mathfrak{g})_{\bar{1}} \subset [\mathfrak{g}, \mathfrak{g}]_{\bar{1}}$ . In other words,  $\mathcal{Z}(\mathfrak{g})_{\bar{0}}$  and  $\mathcal{Z}(\mathfrak{g})_{\bar{1}}$  are totally isotropic subspaces of  $\mathfrak{g}_{\bar{0}}$  and  $\mathfrak{g}_{\bar{1}}$ , respectively. Rewrite  $\mathcal{Z}(\mathfrak{g})_{\bar{0}} = \mathbb{C}X_{\bar{0}} \oplus \mathfrak{l}_{\bar{0}}$ . Then there exist totally isotropic subspaces  $\mathfrak{u}_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}$  of  $\mathfrak{g}_{\bar{0}}$  and  $\mathfrak{u}_{\bar{1}}$  of  $\mathfrak{g}_{\bar{1}}$  such that  $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}, \mathfrak{g}]_{\bar{0}} \oplus (\mathfrak{u}_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}})$ ,  $\mathfrak{g}_{\bar{1}} = [\mathfrak{g}, \mathfrak{g}]_{\bar{1}} \oplus \mathfrak{u}_{\bar{1}}$ , the subspaces  $\mathcal{Z}(\mathfrak{g})_{\bar{0}} \oplus (\mathfrak{u}_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}})$  and  $\mathcal{Z}(\mathfrak{g})_{\bar{1}} \oplus \mathfrak{u}_{\bar{1}}$  are non-degenerate. Let us define

$$Z : \mathfrak{u}_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}} \oplus \mathfrak{u}_{\bar{1}} \rightarrow \mathfrak{l}_{\bar{0}} \oplus \mathbb{C}X_{\bar{0}} \oplus \mathcal{Z}(\mathfrak{g})_{\bar{1}}$$

by: set bases  $\{X_1 = X_{\bar{0}}, X_2, \dots, X_r\}$  of  $\mathfrak{l}_{\bar{0}} \oplus \mathbb{C}X_{\bar{0}}$ ,  $\{Y_1, \dots, Y_t\}$  of  $\mathcal{Z}(\mathfrak{g})_{\bar{1}}$ ,  $\{X'_1 = Y_{\bar{0}}, X'_2, \dots, X'_r\}$  of  $\mathfrak{u}_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}$  and  $\{Y'_1, \dots, Y'_t\}$  of  $\mathfrak{u}_{\bar{1}}$  such that  $B(X_i, X'_j) = \delta_{ij}$ ,  $B(Y_k, Y'_l) = \delta_{kl}$ . Then the map  $Z$  is completely defined by

$$Z \left( \sum_{j=1}^r x_j X'_j \right) = \sum_{i=1}^r \left( \sum_{j=1}^r \mu_{ij} x_j \right) X_i,$$

$$Z \left( \sum_{j=1}^t y_j Y'_j \right) = \sum_{i=1}^t \left( \sum_{j=1}^t \nu_{ij} y_j \right) Y_i$$

with  $\mu_{ij} = \mu_{ji} = B(X'_i, Z(X'_j))$  and  $\nu_{ij} = -\nu_{ji} = B(Y'_i, Z(Y'_j))$ .

It results that the quadratic dimension of  $\mathfrak{g}$  can be calculated as follows:

$$d_q(\mathfrak{g}) = 1 + \frac{\dim(\mathcal{Z}(\mathfrak{g})_{\bar{0}})(1 + \dim(\mathcal{Z}(\mathfrak{g})_{\bar{0}}))}{2} + \frac{\dim(\mathcal{Z}(\mathfrak{g})_{\bar{1}})(\dim(\mathcal{Z}(\mathfrak{g})_{\bar{1}}) - 1)}{2}.$$



## 6. QUASI-SINGULAR QUADRATIC LIE ALGEBRAS

By Definition 5.3, it is natural to question: let  $(\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}, B_{\mathfrak{q}})$  be a quadratic  $\mathbb{Z}_2$ -graded vector space and  $\bar{C}$  be an endomorphism of  $\mathfrak{q}$ . Let  $(\mathfrak{t} = \text{span}\{X_{\bar{1}}, Y_{\bar{1}}\}, B_{\mathfrak{t}})$  be a 2-dimensional *symplectic* vector space with  $B_{\mathfrak{t}}(X_{\bar{1}}, Y_{\bar{1}}) = 1$ . Is there an extension  $\mathfrak{g} = \mathfrak{q} \oplus^{\perp} \mathfrak{t}$  such that  $\mathfrak{g}$  equipped with the bilinear form  $B = B_{\mathfrak{q}} + B_{\mathfrak{t}}$  becomes a quadratic Lie superalgebra such that  $\mathfrak{g}_{\bar{0}} = \mathfrak{q}_{\bar{0}}$ ,  $\mathfrak{g}_{\bar{1}} = \mathfrak{q}_{\bar{1}} \oplus \mathfrak{t}$  and the Lie super-bracket is represented by  $\bar{C}$ ? In this section, we will give an affirmative answer to this question.

The dup-number and the form of the associated invariant  $I$  in the previous sections suggest that it would be also interesting to study a quadratic Lie superalgebra  $\mathfrak{g}$  whose associated invariant  $I$  has the form

$$I = J \wedge p$$

where  $p \in \mathfrak{g}_{\bar{1}}^*$  is nonzero,  $J \in \text{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^1(\mathfrak{g}_{\bar{1}})$  is indecomposable. We obtain the first result as follows:

**Proposition 6.1.**  $\{J, J\} = \{p, J\} = 0$ .

*Proof.* Apply Proposition 1.2 to obtain

$$\begin{aligned} \{I, I\} &= \{J \wedge p, J \wedge p\} = \{J \wedge p, J\} \wedge p + J \wedge \{J \wedge p, p\} \\ &= -\{J, J\} \wedge p \wedge p + 2J \wedge \{p, J\} \wedge p - J \wedge J \wedge \{p, p\}. \end{aligned}$$

Since the super-exterior product is commutative then one has  $J \wedge J = 0$ . Moreover,  $\{I, I\} = 0$  implies that:

$$\{J, J\} \wedge p \wedge p = 2J \wedge \{p, J\} \wedge p.$$

That means  $\{J, J\} \wedge p = 2J \wedge \{p, J\}$ .

If  $\{J, J\} \neq 0$  then  $\{J, J\} \wedge p \neq 0$ , so  $J \wedge \{p, J\} \neq 0$ . Note that  $\{p, J\} \in \text{Alt}^1(\mathfrak{g}_{\bar{0}})$  so  $J$  must contain the factor  $p$ , i.e.  $J = \alpha \otimes p$  where  $\alpha \in \mathfrak{g}_{\bar{0}}^*$ . But  $\{p, J\} = \{p, \alpha \otimes p\} = -\alpha \otimes \{p, p\} = 0$  since  $\{p, p\} = 0$ . This is a contradiction and therefore  $\{J, J\} = 0$ .

As a consequence,  $J \wedge \{p, J\} = 0$ . Set  $\alpha = \{p, J\} \in \text{Alt}^1(\mathfrak{g}_{\bar{0}})$  then we have  $J \wedge \alpha = 0$ . If  $\alpha \neq 0$  then  $J$  must have the form  $J = \alpha \otimes q$  where  $q \in \text{Sym}^1(\mathfrak{g}_{\bar{1}})$ . That is a contradiction since  $J$  is indecomposable.  $\square$

**Definition 6.2.** We continue to keep the condition  $I = J \wedge p$  with  $p \in \mathfrak{g}_{\bar{1}}^*$  nonzero and  $J \in \text{Alt}^1(\mathfrak{g}_{\bar{0}}) \otimes \text{Sym}^1(\mathfrak{g}_{\bar{1}})$  indecomposable. We can assume that

$$J = \sum_{i=1}^n \alpha_i \otimes p_i$$

where  $\alpha_i \in \text{Alt}^1(\mathfrak{g}_{\bar{0}})$ ,  $i = 1, \dots, n$  are linearly independent and  $p_i \in \text{Sym}^1(\mathfrak{g}_{\bar{1}})$ . A quadratic Lie superalgebra having such associated invariant  $I$  is called a *quasi-singular quadratic Lie superalgebra*.

Let  $U = \text{span}\{\alpha_1, \dots, \alpha_n\}$  and  $V = \text{span}\{p_1, \dots, p_n\}$ , one has  $\dim(U)$  and  $\dim(V)$  more than 1 by if there is a contrary then  $J$  is decomposable. Using Definition 1.1, we have:

$$\{J, J\} = \left\{ \sum_{i=1}^n \alpha_i \otimes p_i, \sum_{i=1}^n \alpha_i \otimes p_i \right\} = - \sum_{i,j=1}^n (\{\alpha_i, \alpha_j\} \otimes p_i p_j + (\alpha_i \wedge \alpha_j) \otimes \{p_i, p_j\}).$$

Since  $\{J, J\} = 0$  and  $\alpha_i, i = 1, \dots, n$  are linearly independent then  $\{p_i, p_j\} = 0$ , for all  $i, j$ . It implies that  $\{p_i, J\} = 0$ , for all  $i$ .

Moreover, since  $\{p, J\} = 0$  we obtain  $\{p, p_i\} = 0$ , consequently  $\{p_i, I\} = 0$ , for all  $i$  and  $\{p, I\} = 0$ . By Corollary 1.7 (2) and Lemma 1.14 we conclude that  $\phi^{-1}(V + \mathbb{C}p)$  is a subspace of  $\mathcal{Z}(\mathfrak{g})$  and totally isotropic.

Now, let  $\{q_1, \dots, q_m\}$  be a basis of  $V$  then  $J$  can be rewritten by

$$J = \sum_{j=1}^m \beta_j \otimes q_j$$

where  $\beta_j \in U$ , for all  $j$ . One has:

$$\{J, J\} = \left\{ \sum_{j=1}^m \beta_j \otimes q_j, \sum_{j=1}^m \beta_j \otimes q_j \right\} = - \sum_{i,j=1}^m (\{\beta_i, \beta_j\} \otimes q_i q_j + (\beta_i \wedge \beta_j) \otimes \{q_i, q_j\}).$$

By the linear independence of the system  $\{q_i q_j\}$ , we obtain  $\{\beta_i, \beta_j\} = 0$ , for all  $i, j$ . It implies that  $\{\beta_j, I\} = 0$ , equivalently  $\phi^{-1}(\beta_j) \in \mathcal{Z}(\mathfrak{g})$ , for all  $j$ . Therefore, we always can begin with  $J = \sum_{i=1}^n \alpha_i \otimes p_i$  satisfying the following conditions:

- (i)  $\alpha_i, i = 1, \dots, n$  are linearly independent,
- (ii)  $\phi^{-1}(U)$  and  $\phi^{-1}(V + \mathbb{C}p)$  are totally isotropic subspaces of  $\mathcal{Z}(\mathfrak{g})$  where  $U = \text{span}\{\alpha_1, \dots, \alpha_n\}$  and  $V = \text{span}\{p_1, \dots, p_n\}$ .

Let  $X_0^i = \phi^{-1}(\alpha_i)$ ,  $X_1^i = \phi^{-1}(p_i)$ , for all  $i$  and  $C : \mathfrak{g} \rightarrow \mathfrak{g}$  defined by

$$J(X, Y) = B(C(X), Y), \forall X, Y \in \mathfrak{g}.$$

**Lemma 6.3.** *The mapping  $C$  is a skew-supersymmetric homogeneous endomorphism of odd degree and  $\text{Im}(C) \subset \mathcal{Z}(\mathfrak{g})$ . Recall that if  $C$  is a homogeneous endomorphism of degree  $c$  of  $\mathfrak{g}$  satisfying*

$$B(C(X), Y) = -(-1)^{cx} B(X, C(Y)), \forall X \in \mathfrak{g}_x, Y \in \mathfrak{g}$$

*then we say  $C$  skew-supersymmetric (with respect to  $B$ ).*

*Proof.* Since  $J(\mathfrak{g}_0, \mathfrak{g}_0) = J(\mathfrak{g}_1, \mathfrak{g}_1) = 0$  and  $B$  is even then  $C(\mathfrak{g}_0) \subset \mathfrak{g}_1$  and  $C(\mathfrak{g}_1) \subset \mathfrak{g}_0$ . That means  $C$  is of odd degree. For all  $X \in \mathfrak{g}_0, Y \in \mathfrak{g}_1$  one has:

$$B(C(X), Y) = J(X, Y) = \sum_{i=1}^n \alpha_i \otimes p_i(X, Y) = \sum_{i=1}^n \alpha_i(X) p_i(Y) = \sum_{i=1}^n B(X_0^i, X) B(X_1^i, Y).$$

By the non-degeneracy of  $B$  and  $J(X, Y) = -J(Y, X)$ , we obtain:

$$C(X) = \sum_{i=1}^n B(X_0^i, X) X_1^i \quad \text{and} \quad C(Y) = - \sum_{i=1}^n B(X_1^i, Y) X_0^i.$$

Combined with  $B$  supersymmetric, one has:

$$-B(Y, C(X)) = B(C(X), Y) = -B(C(Y), X) = -B(X, C(Y)).$$

It shows that  $C$  is skew-supersymmetric. Finally,  $\text{Im}(C) \subset \mathcal{Z}(\mathfrak{g})$  since  $X_{\bar{0}}^i, X_{\bar{1}}^i \in \mathcal{Z}(\mathfrak{g})$ , for all  $i$ .  $\square$

**Proposition 6.4.** *Let  $X_{\bar{1}} = \phi^{-1}(p)$  then for all  $X \in \mathfrak{g}_{\bar{0}}, Y, Z \in \mathfrak{g}_{\bar{1}}$  one has:*

- (1)  $[X, Y] = -B(C(X), Y)X_{\bar{1}} - B(X_{\bar{1}}, Y)C(X)$ ,
- (2)  $[Y, Z] = B(X_{\bar{1}}, Y)C(Z) + B(X_{\bar{1}}, Z)C(Y)$ ,
- (3)  $X_{\bar{1}} \in \mathcal{Z}(\mathfrak{g})$  and  $C(X_{\bar{1}}) = 0$ .

*Proof.* Let  $X \in \mathfrak{g}_{\bar{0}}, Y, Z \in \mathfrak{g}_{\bar{1}}$  then

$$\begin{aligned} B([X, Y], Z) &= J \wedge p(X, Y, Z) = -J(X, Y)p(Z) - J(X, Z)p(Y) \\ &= -B(C(X), Y)B(X_{\bar{1}}, Z) - B(C(X), Z)B(X_{\bar{1}}, Y). \end{aligned}$$

By the non-degeneracy of  $B$  on  $\mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}}$ , it shows that:

$$[X, Y] = -B(C(X), Y)X_{\bar{1}} - B(X_{\bar{1}}, Y)C(X).$$

Combined with  $B$  invariant and  $C$  skew-supersymmetric, one has:

$$[Y, Z] = B(X_{\bar{1}}, Y)C(Z) + B(X_{\bar{1}}, Z)C(Y).$$

Since  $\{p, I\} = 0$  then  $X_{\bar{1}} \in \mathcal{Z}(\mathfrak{g})$ . Moreover,  $\{p, p_i\} = 0$  imply  $B(X_{\bar{1}}, X_{\bar{1}}^i) = 0$ , for all  $i$ . It means  $B(X_{\bar{1}}, \text{Im}(C)) = 0$ . And since  $B(C(X_{\bar{1}}), X) = B(X_{\bar{1}}, C(X)) = 0$ , for all  $X \in \mathfrak{g}$  then  $C(X_{\bar{1}}) = 0$ .  $\square$

Let  $W$  be a complementary subspace of  $\text{span}\{X_{\bar{1}}^1, \dots, X_{\bar{1}}^n, X_{\bar{1}}\}$  in  $\mathfrak{g}_{\bar{1}}$  and  $Y_{\bar{1}}$  be an element in  $W$  such that  $B(X_{\bar{1}}, Y_{\bar{1}}) = 1$ . Let  $X_{\bar{0}} = C(Y_{\bar{1}})$ ,  $\mathfrak{q} = (\mathbb{C}X_{\bar{1}} \oplus \mathbb{C}Y_{\bar{1}})^\perp$  and  $B_{\mathfrak{q}} = B|_{\mathfrak{q} \times \mathfrak{q}}$  then we have the following corollary:

**Corollary 6.5.**

- (1)  $[Y_{\bar{1}}, Y_{\bar{1}}] = 2X_{\bar{0}}, [Y_{\bar{1}}, X] = C(X) - B(X, X_{\bar{0}})X_{\bar{1}}$  and  $[X, Y] = -B(C(X), Y)X_{\bar{1}}$ , for all  $X, Y \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$ .
- (2)  $[\mathfrak{g}, \mathfrak{g}] \subset \text{Im}(C) + \mathbb{C}X_{\bar{1}} \subset \mathcal{Z}(\mathfrak{g})$  so  $\mathfrak{g}$  is 2-step nilpotent. If  $\mathfrak{g}$  is reduced then  $[\mathfrak{g}, \mathfrak{g}] = \text{Im}(C) + \mathbb{C}X_{\bar{1}} = \mathcal{Z}(\mathfrak{g})$ .
- (3)  $C^2 = 0$ .

*Proof.*

- (1) The assertion (1) is obvious by Proposition 6.4.
- (2) Note that  $X_{\bar{0}} \in \text{Im}(C)$  so  $[\mathfrak{g}, \mathfrak{g}] \subset \text{Im}(C) + \mathbb{C}X_{\bar{1}}$ . By Lemma 6.3 and Proposition 6.4,  $\text{Im}(C) + \mathbb{C}X_{\bar{1}} \subset \mathcal{Z}(\mathfrak{g})$ . If  $\mathfrak{g}$  is reduced then  $\mathcal{Z}(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$  and therefore  $[\mathfrak{g}, \mathfrak{g}] = \text{Im}(C) + \mathbb{C}X_{\bar{1}} = \mathcal{Z}(\mathfrak{g})$ .
- (3) Since  $\mathfrak{g}$  is 2-step nilpotent then

$$0 = [Y_{\bar{1}}, [Y_{\bar{1}}, Y_{\bar{1}}]] = [Y_{\bar{1}}, 2X_{\bar{0}}] = 2C(X_{\bar{0}}) - 2B(X_{\bar{0}}, X_{\bar{0}})X_{\bar{1}}.$$

Since  $X_{\bar{0}} = C(Y_{\bar{1}})$  and  $\text{Im}(C)$  is totally isotropic then  $B(X_{\bar{0}}, X_{\bar{0}}) = 0$  and therefore  $C(X_{\bar{0}}) = C^2(Y_{\bar{1}}) = 0$ .

If  $X \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$  then  $0 = [Y_{\bar{1}}, [Y_{\bar{1}}, X]] = [Y_{\bar{1}}, C(X)]$ . By the choice of  $Y_{\bar{1}}$ , it is sure that  $C(X) \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$ . Therefore, one has:

$$0 = [Y_{\bar{1}}, C(X)] = C^2(X) - B(C(X), X_{\bar{0}})X_{\bar{1}} = C^2(X) - B(C(X), C(Y_{\bar{1}}))X_{\bar{1}}.$$

By  $\text{Im}(C)$  totally isotropic, one has  $C^2(X) = 0$ .

□

Now, we consider a special case:  $X_{\bar{0}} = 0$ . As a consequence,  $[Y_{\bar{1}}, Y_{\bar{1}}] = 0$ ,  $[Y_{\bar{1}}, X] = C(X)$  and  $[X, Y] = -B(C(X), Y)X_{\bar{1}}$ , for all  $X, Y \in \mathfrak{q}$ . Let  $X \in \mathfrak{q}$  and assume that  $C(X) = C_1(X) + aX_1$  where  $C_1(X) \in \mathfrak{q}$  then

$$0 = B([Y_{\bar{1}}, Y_{\bar{1}}], X) = B(Y_{\bar{1}}, [Y_{\bar{1}}, X]) = B(Y_{\bar{1}}, C_1(X) + aX_1) = a.$$

It shows that  $C(X) \in \mathfrak{q}$ , for all  $X \in \mathfrak{q}$  and therefore we have an affirmative answer of the above question as follows:

**Proposition 6.6.** *Let  $(\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}, B_q)$  be a quadratic  $\mathbb{Z}_2$ -graded vector space and  $\bar{C}$  be an odd endomorphism of  $\mathfrak{q}$  such that  $\bar{C}$  is skew-supersymmetric and  $\bar{C}^2 = 0$ . Let  $(\mathfrak{t} = \text{span}\{X_{\bar{1}}, Y_{\bar{1}}\}, B_t)$  be a 2-dimensional symplectic vector space with  $B_t(X_{\bar{1}}, Y_{\bar{1}}) = 1$ .*

1. *Consider the space  $\mathfrak{g} = \mathfrak{q} \oplus^{\perp} \mathfrak{t}$  and define the product on  $\mathfrak{g}$  by:*

$$[Y_{\bar{1}}, Y_{\bar{1}}] = [X_{\bar{1}}, \mathfrak{g}] = 0, [Y_{\bar{1}}, X] = \bar{C}(X) \text{ and } [X, Y] = -B_q(\bar{C}(X), Y)X_{\bar{1}}$$

*for all  $X \in \mathfrak{q}$ . Then  $\mathfrak{g}$  becomes a 2-nilpotent quadratic Lie superalgebra with the bilinear form  $B = B_q + B_t$ . Moreover, one has  $\mathfrak{g}_{\bar{0}} = \mathfrak{q}_{\bar{0}}$ ,  $\mathfrak{g}_{\bar{1}} = \mathfrak{q}_{\bar{1}} \oplus \mathfrak{t}$ .*

*Remark 6.7.* The method above remains valid for the elementary quadratic Lie superalgebra  $\mathfrak{g}_6^s$  with  $I$  decomposable (see Section 3) as follows: let  $\mathfrak{q} = (\mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}) \oplus (\mathbb{C}Z_{\bar{1}} \oplus \mathbb{C}T_{\bar{1}})$  where  $\mathfrak{q}_{\bar{0}} = \text{span}\{X_{\bar{0}}, Y_{\bar{0}}\}$ ,  $\mathfrak{q}_{\bar{1}} = \text{span}\{Z_{\bar{1}}, T_{\bar{1}}\}$  and the bilinear form  $B_q$  is defined by  $B(X_{\bar{0}}, Y_{\bar{0}}) = B(Z_{\bar{1}}, T_{\bar{1}}) = 1$ , the other are zero. Let  $C : \mathfrak{q} \rightarrow \mathfrak{q}$  be a linear map defined by:

$$\bar{C} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then  $C$  is odd and  $C^2 = 0$ . Set the vector space  $\mathfrak{g} = \mathfrak{q} \oplus^{\perp} \mathfrak{t}$ , where  $(\mathfrak{t} = \text{span}\{X_{\bar{1}}, Y_{\bar{1}}\}, B_t)$  is a 2-dimensional symplectic vector space with  $B_t(X_{\bar{1}}, Y_{\bar{1}}) = 1$ . Then  $\mathfrak{g} = \mathfrak{g}_6^s$  with the Lie super-bracket defined as in Proposition 6.6.

It remains to consider  $X_{\bar{0}} \neq 0$ . The fact is that  $C$  may be not stable on  $\mathfrak{q}$ , that is,  $C(X) \in \mathfrak{q} \oplus \mathbb{C}X_{\bar{1}}$  if  $X \in \mathfrak{q}$  but that we need here is an action stable on  $\mathfrak{q}$ . Therefore, we decompose  $C$  by  $C(X) = \bar{C}(X) + \varphi(X)X_{\bar{1}}$ , for all  $X \in \mathfrak{q}$  where  $\bar{C} : \mathfrak{q} \rightarrow \mathfrak{q}$  and  $\varphi : \mathfrak{q} \rightarrow \mathbb{C}$ . Since  $B(C(Y_{\bar{1}}), X) = B(Y_{\bar{1}}, C(X))$  then  $\varphi(X) = -B(X_{\bar{0}}, X) = -B(X, X_{\bar{0}})$ , for all  $X \in \mathfrak{q}$ . Moreover,  $C$  is odd degree on  $\mathfrak{g}$  and skew-supersymmetric (with respect to  $B$ ) implies that  $\bar{C}$  is also odd on  $\mathfrak{q}$  and skew-supersymmetric (with respect to  $B_q$ ). It is easy to see that  $\bar{C}^2 = 0$ ,  $\bar{C}(X_{\bar{0}}) = 0$  and we have the following result:

**Corollary 6.8.** *Keep the notations as in Corollary 6.5 and replace  $2X_{\bar{0}}$  by  $X_{\bar{0}}$  then for all  $X, Y \in \mathfrak{q}$ , one has:*

- $[Y_{\bar{1}}, Y_{\bar{1}}] = X_{\bar{0}},$
- $[Y_{\bar{1}}, X] = \bar{C}(X) - B(X, X_{\bar{0}})X_{\bar{1}},$
- $[X, Y] = -B(\bar{C}(X), Y)X_{\bar{1}}.$

Hence, we have a more general result of Proposition 6.6:

**Proposition 6.9.** *Let  $(\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}, B_{\mathfrak{q}})$  be a quadratic  $\mathbb{Z}_2$ -graded vector space and  $\bar{C}$  an odd endomorphism of  $\mathfrak{q}$  such that  $\bar{C}$  is skew-supersymmetric and  $\bar{C}^2 = 0$ . Let  $X_{\bar{0}}$  be an isotropic element of  $\mathfrak{q}_{\bar{0}}$ ,  $X_{\bar{0}} \in \ker(\bar{C})$  and  $(\mathfrak{t} = \text{span}\{X_{\bar{1}}, Y_{\bar{1}}\}, B_{\mathfrak{t}})$  be a 2-dimensional symplectic vector space with  $B_{\mathfrak{t}}(X_{\bar{1}}, Y_{\bar{1}}) = 1$ . Consider the space  $\mathfrak{g} = \mathfrak{q} \oplus^{\perp} \mathfrak{t}$  and define the product on  $\mathfrak{g}$  by:*

$$[Y_{\bar{1}}, Y_{\bar{1}}] = X_{\bar{0}}, [Y_{\bar{1}}, X] = \bar{C}(X) - B_{\mathfrak{q}}(X, X_{\bar{0}})X_{\bar{1}} \text{ and } [X, Y] = -B_{\mathfrak{q}}(\bar{C}(X), Y)X_{\bar{1}}$$

*for all  $X \in \mathfrak{q}$ . Then  $\mathfrak{g}$  becomes a 2-nilpotent quadratic Lie superalgebra with the bilinear form  $B = B_{\mathfrak{q}} + B_{\mathfrak{t}}$ . Moreover, one has  $\mathfrak{g}_{\bar{0}} = \mathfrak{q}_{\bar{0}}$ ,  $\mathfrak{g}_{\bar{1}} = \mathfrak{q}_{\bar{1}} \oplus \mathfrak{t}$ .*

A quadratic Lie superalgebra obtained in the above proposition is a special case of the generalized double extensions given in [BBB] where the authors consider the generalized double extension of a quadratic  $\mathbb{Z}_2$ -graded vector space (regarded as an Abelian superalgebra) by a one-dimensional Lie superalgebra.

## 7. APPENDIX: ADJOINT ORBITS OF $\mathfrak{sp}(2n)$ AND $\mathfrak{o}(m)$

This appendix recalls a fundamental and really interesting problem in Lie theory that is necessary for the paper: the classification of adjoint orbits of classical Lie algebras  $\mathfrak{sp}(2n)$  and  $\mathfrak{o}(m)$  where  $m, n \in \mathbb{N}^*$ . A brief overview can be found in [Hum95] with interesting discussions. Many results with detailed proofs can be found in [CM93].

A different point here is to use the Fitting decomposition to review this problem. In particular, we parametrize the *invertible* component in the Fitting decomposition of a skew-symmetric map and from this, we give an explicit classification for  $\text{Sp}(2n)$ -adjoint orbits of  $\mathfrak{sp}(2n)$  and  $\text{O}(m)$ -adjoint orbits of  $\mathfrak{o}(m)$  in the general case. In other words, we establish a one-to-one correspondence between the set of orbits and some set of indices. This is an rather obvious and classical result but in our knowledge there is not a reference for that mentioned before.

Let  $V$  be a  $m$ -dimensional complex vector space endowed with a non-degenerate bilinear form  $B_{\varepsilon}$  where  $\varepsilon = \pm 1$  such that  $B_{\varepsilon}(X, Y) = \varepsilon B_{\varepsilon}(Y, X)$ , for all  $X, Y \in V$ . If  $\varepsilon = 1$  then the form  $B_1$  is symmetric and we say  $V$  a *quadratic* vector space. If  $\varepsilon = -1$  then  $m$  must be even and we say  $V$  a *symplectic* vector space with symplectic form  $B_{-1}$ . We denote by  $\mathcal{L}(V)$  the algebra of linear operators of  $V$  and by  $\text{GL}(V)$  the group of invertible operators in  $\mathcal{L}(V)$ . A map  $C \in \mathcal{L}(V)$  is called *skew-symmetric* (with respect to  $B_{\varepsilon}$ ) if it satisfies the following condition:

$$B_{\varepsilon}(C(X), Y) = -B_{\varepsilon}(X, C(Y)), \forall X, Y \in V.$$

We define

$$I_\varepsilon(V) = \{A \in \text{GL}(V) \mid B_\varepsilon(A(X), A(Y)) = B_\varepsilon(X, Y), \forall X, Y \in V\}$$

$$\text{and } \mathfrak{g}_\varepsilon(V) = \{C \in \mathcal{L}(V) \mid C \text{ is skew-symmetric}\}.$$

Then  $I_\varepsilon(V)$  is the *isometry group* of the bilinear form  $B_\varepsilon$  and  $\mathfrak{g}_\varepsilon(V)$  is its Lie algebra. Denote by  $A^* \in \mathcal{L}(V)$  the *adjoint map* of an element  $A \in \mathcal{L}(V)$  with respect to  $B_\varepsilon$ , then  $A \in I_\varepsilon(V)$  if and only if  $A^{-1} = A^*$  and  $C \in \mathfrak{g}_\varepsilon(V)$  if and only if  $C^* = -C$ . If  $\varepsilon = 1$  then  $I_\varepsilon(V)$  is denoted by  $\text{O}(V)$  and  $\mathfrak{g}_\varepsilon(V)$  is denoted by  $\mathfrak{o}(V)$ . If  $\varepsilon = -1$  then  $\text{Sp}(V)$  stands for  $I_\varepsilon(V)$  and  $\mathfrak{sp}(V)$  stands for  $\mathfrak{g}_\varepsilon(V)$ .

Recall that the *adjoint action*  $\text{Ad}$  of  $I_\varepsilon(V)$  on  $\mathfrak{g}_\varepsilon(V)$  is given by

$$\text{Ad}_U(C) = UCU^{-1}, \forall U \in I_\varepsilon(V), C \in \mathfrak{g}_\varepsilon(V).$$

We denote by  $\mathcal{O}_C = \text{Ad}_{I_\varepsilon(V)}(C)$ , the *adjoint orbit* of an element  $C \in \mathfrak{g}_\varepsilon(V)$  by this action.

If  $V = \mathbb{C}^m$ , we call  $B_\varepsilon$  a *canonical bilinear form* of  $\mathbb{C}^m$ . And with respect to  $B_\varepsilon$ , we define a *canonical basis*  $\mathcal{B} = \{E_1, \dots, E_m\}$  of  $\mathbb{C}^m$  as follows. If  $m$  even,  $m = 2n$ , write  $\mathcal{B} = \{E_1, \dots, E_n, F_1, \dots, F_n\}$ , if  $m$  is odd,  $m = 2n + 1$ , write  $\mathcal{B} = \{E_1, \dots, E_n, G, F_1, \dots, F_n\}$  and one has:

- if  $m = 2n$  then

$$B_1(E_i, F_j) = B_1(F_j, E_i) = \delta_{ij}, B_1(E_i, E_j) = B_1(F_i, F_j) = 0,$$

$$B_{-1}(E_i, F_j) = -B_{-1}(F_j, E_i) = \delta_{ij}, B_{-1}(E_i, E_j) = B_{-1}(F_i, F_j) = 0,$$

where  $1 \leq i, j \leq n$ .

In the case of  $\varepsilon = -1$ ,  $\mathcal{B}$  is also called a *Darboux basis* of  $\mathbb{C}^{2n}$ .

- if  $m = 2n + 1$  then  $\varepsilon = 1$  and

$$\begin{cases} B_1(E_i, F_j) = \delta_{ij}, B_1(E_i, E_j) = B_1(F_i, F_j) = 0, \\ B_1(E_i, G) = B_1(F_j, G) = 0, \\ B_1(G, G) = 1 \end{cases}$$

where  $1 \leq i, j \leq n$ .

Also, in the case  $V = \mathbb{C}^m$ , we denote by  $\text{GL}(m)$  instead of  $\text{GL}(V)$ ,  $\text{O}(m)$  stands for  $\text{O}(V)$  and  $\mathfrak{o}(m)$  stands for  $\mathfrak{o}(V)$ . If  $m = 2n$  then  $\text{Sp}(2n)$  stands for  $\text{Sp}(V)$  and  $\mathfrak{sp}(2n)$  stands for  $\mathfrak{sp}(V)$ . We will also write  $I_\varepsilon = I_\varepsilon(\mathbb{C}^m)$  and  $\mathfrak{g}_\varepsilon = \mathfrak{g}_\varepsilon(\mathbb{C}^m)$ . Our goal is classifying all of  $I_\varepsilon$ -adjoint orbits of  $\mathfrak{g}_\varepsilon$ .

Finally, let  $V$  is an  $m$ -dimensional vector space. If  $V$  is quadratic then  $V$  is isometrically isomorphic to the quadratic space  $(\mathbb{C}^m, B_1)$  and if  $V$  is symplectic then  $V$  is isometrically isomorphic to the symplectic space  $(\mathbb{C}^m, B_{-1})$  [Bou59].

### 7.1. Nilpotent orbits.

Let  $n \in \mathbb{N}^*$ , a *partition*  $[d]$  of  $n$  is a tuple  $[d_1, \dots, d_k]$  of positive integers satisfying

$$d_1 \geq \dots \geq d_k \text{ and } d_1 + \dots + d_k = n.$$

Occasionally, we use the notation  $[t_1^{i_1}, \dots, t_r^{i_r}]$  to replace the partition  $[d_1, \dots, d_k]$  where

$$d_j = \begin{cases} t_1 & 1 \leq j \leq i_1 \\ t_2 & i_1 + 1 \leq j \leq i_1 + i_2 \\ t_3 & i_1 + i_2 + 1 \leq j \leq i_1 + i_2 + i_3 \\ \dots & \dots \end{cases}$$

Each  $i_j$  is called the *multiplicity* of  $t_j$ . Denote by  $\mathcal{P}(n)$  the set of partitions of  $n$ .

Let  $p \in \mathbb{N}^*$ . We denote the *Jordan block of size  $p$*  by  $J_1 = (0)$  and for  $p \geq 2$ ,

$$J_p := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then  $J_p$  is a nilpotent endomorphism of  $\mathbb{C}^p$ . Given a partition  $[d] = [d_1, \dots, d_k] \in \mathcal{P}(n)$  there is a nilpotent endomorphism of  $\mathbb{C}^n$  defined by

$$X_{[d]} := \text{diag}_k(J_{d_1}, \dots, J_{d_k}).$$

Moreover,  $X_{[d]}$  is also a nilpotent element of  $\mathfrak{sl}(n)$  since its trace is zero. Conversely, if  $C$  is a nilpotent element in  $\mathfrak{sl}(n)$  then  $C$  is  $\text{GL}(n)$ -conjugate to its *Jordan normal form*  $X_{[d]}$  for some partition  $[d] \in \mathcal{P}(n)$ .

Given two different partitions  $[d] = [d_1, \dots, d_k]$  and  $[d'] = [d'_1, \dots, d'_l]$  of  $n$  then the  $\text{GL}(n)$ -adjoint orbits through  $X_{[d]}$  and  $X_{[d']}$  respectively are disjoint by the unicity of Jordan normal form. Therefore, one has the following proposition:

**Proposition 7.1.** *There is a one-to-one correspondence between the set of nilpotent  $\text{GL}(n)$ -adjoint orbits of  $\mathfrak{sl}(n)$  and the set  $\mathcal{P}(n)$ .*

Define the set

$$\mathcal{P}_\varepsilon(m) = \{[d_1, \dots, d_m] \in \mathcal{P}(m) \mid \#\{j \mid d_j = i\} \text{ is even for all } i \text{ such that } (-1)^i = \varepsilon\}.$$

In particular,  $\mathcal{P}_1(m)$  is the set of partitions of  $m$  in which even parts occur with even multiplicity and  $\mathcal{P}_{-1}(m)$  is the set of partitions of  $m$  in which odd parts occur with even multiplicity.

**Proposition 7.2** (Gerstenhaber).

*Nilpotent  $I_\varepsilon$ -adjoint orbits in  $\mathfrak{g}_\varepsilon$  are in one-to-one correspondence with the set of partitions in  $\mathcal{P}_\varepsilon(m)$ .*

Here, we give a construction of a nilpotent element in  $\mathfrak{g}_\varepsilon$  from a partition  $[d]$  of  $m$  that is useful for this paper. Define maps in  $\mathfrak{g}_\varepsilon$  as follows:

- For  $p \geq 2$ , we equip the vector space  $\mathbb{C}^{2p}$  with its canonical bilinear form  $B_\varepsilon$  and the map  $C_{2p}^J$  having the matrix

$$C_{2p}^J = \begin{pmatrix} J_p & 0 \\ 0 & -{}^t J_p \end{pmatrix}$$



in a canonical basis where  ${}^t J_p$  denotes the *transpose* matrix of the Jordan block  $J_p$ . Then  $C_{2p}^J \in \mathfrak{g}_\varepsilon(\mathbb{C}^{2p})$ .

- For  $p \geq 1$  we equip the vector space  $\mathbb{C}^{2p+1}$  with its canonical bilinear form  $B_1$  and the map  $C_{2p+1}^J$  having the matrix

$$C_{2p+1}^J = \begin{pmatrix} J_{p+1} & M \\ 0 & -{}^t J_p \end{pmatrix}$$

in a canonical basis where  $M = (m_{ij})$  denotes the  $(p+1) \times p$ -matrix with  $m_{p+1,p} = -1$  and  $m_{ij} = 0$  otherwise. Then  $C_{2p+1}^J \in \mathfrak{o}(2p+1)$

- For  $p \geq 1$ , we consider the vector space  $\mathbb{C}^{2p}$  equipped with its canonical bilinear form  $B_{-1}$  and the map  $C_{p+p}^J$  with matrix

$$\begin{pmatrix} J_p & M \\ 0 & -{}^t J_p \end{pmatrix}$$

in a canonical basis where  $M = (m_{ij})$  denotes the  $p \times p$ -matrix with  $m_{p,p} = 1$  and  $m_{ij} = 0$  otherwise. Then  $C_{p+p}^J \in \mathfrak{sp}(2p)$ .

For each partition  $[d] \in \mathcal{P}_{-1}(2n)$ ,  $[d]$  can be written as

$$(p_1, p_1, p_2, p_2, \dots, p_k, p_k, 2q_1, \dots, 2q_\ell)$$

with all  $p_i$  odd,  $p_1 \geq p_2 \geq \dots \geq p_k$  and  $q_1 \geq q_2 \geq \dots \geq q_\ell$ . We associate  $[d]$  to the map  $C_{[d]}$  with matrix:

$$\text{diag}_{k+\ell}(C_{2p_1}^J, C_{2p_2}^J, \dots, C_{2p_k}^J, C_{q_1+q_1}^J, \dots, C_{q_\ell+q_\ell}^J)$$

in a canonical basis of  $\mathbb{C}^{2n}$  then  $C_{[d]} \in \mathfrak{sp}(2n)$ .

Similarly, let  $[d] \in \mathcal{P}_1(m)$ ,  $[d]$  can be written as

$$(p_1, p_1, p_2, p_2, \dots, p_k, p_k, 2q_1 + 1, \dots, 2q_\ell + 1)$$

with all  $p_i$  even,  $p_1 \geq p_2 \geq \dots \geq p_k$  and  $q_1 \geq q_2 \geq \dots \geq q_\ell$ . We associate  $[d]$  to the map  $C_{[d]}$  with matrix:

$$\text{diag}_{k+\ell}(C_{2p_1}^J, C_{2p_2}^J, \dots, C_{2p_k}^J, C_{2q_1+1}^J, \dots, C_{2q_\ell+1}^J).$$

in a canonical basis of  $\mathbb{C}^m$  then  $C_{[d]} \in \mathfrak{o}(m)$ .

By Proposition 7.2, it is sure that our construction is a bijection between the set  $\mathcal{P}_\varepsilon(m)$  and the set of nilpotent  $I_\varepsilon$ -adjoint orbits in  $\mathfrak{g}_\varepsilon$ .

## 7.2. Semisimple orbits.

We recall a well-known result [CM93]:

**Proposition 7.3.** *Let  $\mathfrak{g}$  be a semisimple Lie algebra,  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$  and  $W$  be the associated Weyl group. Then there is a bijection between the set of semisimple orbits of  $\mathfrak{g}$  and  $\mathfrak{h}/W$ .*

For each  $\mathfrak{g}_\varepsilon$ , we choose the Cartan subalgebra  $\mathfrak{h}$  given by the vector space of diagonal matrices of type

$$\text{diag}_{2n}(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n)$$

if  $\mathfrak{g}_\varepsilon = \mathfrak{o}(2n)$  or  $\mathfrak{g}_\varepsilon = \mathfrak{sp}(2n)$  and of type

$$\text{diag}_{2n+1}(\lambda_1, \dots, \lambda_n, 0, -\lambda_1, \dots, -\lambda_n)$$

if  $\mathfrak{g}_\varepsilon = \mathfrak{o}(2n+1)$ .

Any diagonalizable (equivalently semisimple)  $C \in \mathfrak{g}_\varepsilon$  is conjugate to an element of  $\mathfrak{h}$ .

If  $\mathfrak{g}_\varepsilon = \mathfrak{sp}(2n)$  then any two eigenvectors  $v, w \in \mathbb{C}^{2n}$  of  $C \in \mathfrak{g}_\varepsilon$  with eigenvalues  $\lambda, \lambda' \in \mathbb{C}$  such that  $\lambda + \lambda' \neq 0$  are orthogonal. Moreover, each eigenvalue pair  $\lambda, -\lambda$  is corresponding to an eigenvector pair  $(v, w)$  satisfying  $B_\varepsilon(v, w) = 1$  and we can easily arrange for vectors  $v, v'$  lying in a distinct pair  $(v, w), (v', w')$  to be orthogonal, regardless of the eigenvalues involved. That means the associated Weyl group is of all coordinate permutations and sign changes of  $(\lambda_1, \dots, \lambda_n)$ . We denote it by  $G_n$ .

If  $\mathfrak{g}_\varepsilon = \mathfrak{o}(2n)$ , the associated Weyl group, when considered in the action of the group  $\text{SO}(2n)$ , consists all coordinate permutations and even sign changes of  $(\lambda_1, \dots, \lambda_n)$ . However, we only focus on  $\text{O}(2n)$ -adjoint orbits of  $\mathfrak{o}(2n)$  obtained by the action of the full orthogonal group, then similarly to preceding analysis any sign change effects. The corresponding group is still  $G_n$ . If  $\mathfrak{g}_\varepsilon = \mathfrak{o}(2n+1)$ , the Weyl group is  $G_n$  and there is nothing to add.

Now, let  $\Lambda_n = \{(\lambda_1, \dots, \lambda_n) \mid \lambda_1, \dots, \lambda_n \in \mathbb{C}, \lambda_i \neq 0 \text{ for some } i\}$ .

**Corollary 7.4.** *There is a bijection between nonzero semisimple  $I_\varepsilon$ -adjoint orbits of  $\mathfrak{g}_\varepsilon$  and  $\Lambda_n/G_n$ .*

### 7.3. Invertible orbits.

**Definition 7.5.** We say that the  $I_\varepsilon$ -orbit  $\mathcal{O}_X$  is *invertible* if  $X$  is an invertible element in  $\mathfrak{g}_\varepsilon$ .

Keep the above notations. We say an element  $X \in V$  *isotropic* if  $B_\varepsilon(X, X) = 0$  and a subset  $W \subset V$  *totally isotropic* if  $B_\varepsilon(X, Y) = 0$  for all  $X, Y \in W$ .

We recall the classification method given in [DPU] as follows. First, we need the lemma:

**Lemma 7.6.** *Let  $V$  be an even-dimensional vector space with a non-degenerate bilinear form  $B_\varepsilon$ . Assume that  $V = V_+ \oplus V_-$  where  $V_\pm$  are totally isotropic vector subspaces.*

- (1) *Let  $N \in \mathcal{L}(V)$  such that  $N(V_\pm) \subset V_\pm$ . We define maps  $N_\pm$  by  $N_+|_{V_+} = N|_{V_+}$ ,  $N_+|_{V_-} = 0$ ,  $N_-|_{V_-} = N|_{V_-}$  and  $N_-|_{V_+} = 0$ . Then  $N \in \mathfrak{g}_\varepsilon(V)$  if and only if  $N_- = -N_+^*$  and, in this case,  $N = N_+ - N_+^*$ .*
- (2) *Let  $U_+ \in \mathcal{L}(V)$  such that  $U_+$  is invertible,  $U_+(V_+) = V_+$  and  $U_+|_{V_-} = \text{Id}_{V_-}$ . We define  $U \in \mathcal{L}(V)$  by  $U|_{V_+} = U_+|_{V_+}$  and  $U|_{V_-} = (U_+^{-1})^*|_{V_-}$ . Then  $U \in I_\varepsilon(V)$ .*
- (3) *Let  $N' \in \mathfrak{g}_\varepsilon(V)$  such that  $N'$  satisfies the assumptions of (1). Define  $N_\pm$  as in (1). Moreover, we assume that there exists  $U_+ \in \mathcal{L}(V_+)$ ,  $U_+$  invertible such that*

$$N'_+|_{V_+} = (U_+ N_+ U_+^{-1})|_{V_+}.$$

We extend  $U_+$  to  $V$  by  $U_+|_{V_-} = \text{Id}_{V_-}$  and define  $U \in I_\varepsilon(V)$  as in (2). Then

$$N' = U N U^{-1}.$$

*Proof.*

- (1) It is obvious that  $N = N_+ + N_-$ . Recall that  $N \in \mathfrak{g}_\varepsilon(V)$  if and only if  $N^* = -N$  so  $N_+^* + N_-^* = -N_+ - N_-$ . Since  $B_\varepsilon(N_+^*(V_+), V) = B_\varepsilon(V_+, N_+(V)) = 0$  then  $N_+^*(V_+) = 0$ . Similarly,  $N_-^*(V_-) = 0$ . Hence,  $N_- = -N_+^*$ .
- (2) We shows that  $B_\varepsilon(U(X), U(Y)) = B_\varepsilon(X, Y)$ , for all  $X, Y \in V$ . Indeed, let  $X = X_+ + X_-$ ,  $Y = Y_+ + Y_- \in V_+ \oplus V_-$ , one has

$$\begin{aligned} B_\varepsilon(U(X_+ + X_-), U(Y_+ + Y_-)) &= B_\varepsilon(U_+(X_+) + (U_+^{-1})^*(X_-), U_+(Y_+) + (U_+^{-1})^*(Y_-)) \\ &= B_\varepsilon(U_+(X_+), (U_+^{-1})^*(Y_-)) + B_\varepsilon((U_+^{-1})^*(X_-), U_+(Y_+)) \\ &= B_\varepsilon(X_+, Y_-) + B_\varepsilon(X_-, Y_+) = B_\varepsilon(X, Y). \end{aligned}$$

- (3) Since  $B_\varepsilon(U^{-1}(V_+), V_+) = B_\varepsilon(V_+, U(V_+)) = 0$ , one has  $U^{-1}(V_+) = V_+$  and  $U^{-1}(V_-) = V_-$ . Consequently,  $(U N U^{-1})(V_+) \subset V_+$  and  $(U N U^{-1})(V_-) \subset V_-$ . Clearly,  $U N U^{-1} \in \mathfrak{g}_\varepsilon(V)$ . By (1), we only show that

$$(U N U^{-1})|_{V_+} = N'_+$$

This is obvious since  $U^{-1}|_{V_+} = U_+^{-1}$ .

□

Let us now consider  $C \in \mathfrak{g}_\varepsilon$ ,  $C$  invertible. Then,  $m$  must be even (obviously, it happened if  $\varepsilon = -1$ ),  $m = 2n$ . Indeed, we assume that  $\varepsilon = 1$  then the skew-symmetric form  $\Delta_C$  on  $\mathbb{C}^m$  defined by  $\Delta_C(v_1, v_2) = B_1(v_1, C(v_2))$  is non-degenerate. and the assertion follows. We decompose  $C = S + N$  into semisimple and nilpotent parts,  $S, N \in \mathfrak{g}_\varepsilon$  by its Jordan decomposition. It is clear that  $S$  is invertible. We have  $\lambda \in \Lambda$  if and only if  $-\lambda \in \Lambda$  where  $\Lambda$  is the spectrum of  $S$ . Also,  $m(\lambda) = m(-\lambda)$ , for all  $\lambda \in \Lambda$  with the multiplicity  $m(\lambda)$ . Since  $N$  and  $S$  commute, we have  $N(V_{\pm\lambda}) \subset V_{\pm\lambda}$  where  $V_\lambda$  is the eigenspace of  $S$  corresponding to  $\lambda \in \Lambda$ . Denote by  $W(\lambda)$  the direct sum

$$W(\lambda) = V_\lambda \oplus V_{-\lambda}.$$

Define the equivalence relation  $\mathcal{R}$  on  $\Lambda$  by:

$$\lambda \mathcal{R} \mu \text{ if and only if } \lambda = \pm \mu.$$

Then

$$\mathbb{C}^{2n} = \bigoplus_{\lambda \in \Lambda/\mathcal{R}}^\perp W(\lambda),$$

and each  $(W(\lambda), B_\lambda)$  is a vector space with the non-degenerate form  $B_\lambda$  given by:

$$B_\lambda = B_\varepsilon|_{W(\lambda) \times W(\lambda)}.$$

Fix  $\lambda \in \Lambda$ . We write  $W(\lambda) = V_+ \oplus V_-$  with  $V_\pm = V_{\pm\lambda}$ . Then, according to the notation in Lemma 7.6, define  $N_{\pm\lambda} = N_\pm$ . Since  $N|_{V_-} = -N_\lambda^*$ , it is easy to verify that the matrices of  $N|_{V_+}$  and  $N|_{V_-}$  have the same Jordan form. Let  $(d_1(\lambda), \dots, d_{r_\lambda}(\lambda))$  be the size of the Jordan blocks in the Jordan decomposition of  $N|_{V_+}$ . This does not

depend on a possible choice between  $N|_{V_+}$  or  $N|_{V_-}$  since both maps have the same Jordan type.

Next, we consider

$$\mathcal{D} = \bigcup_{r \in \mathbb{N}^*} \{(d_1, \dots, d_r) \in \mathbb{N}^r \mid d_1 \geq d_2 \geq \dots \geq d_r \geq 1\}.$$

Define  $d : \Lambda \rightarrow \mathcal{D}$  by  $d(\lambda) = (d_1(\lambda), \dots, d_{r_\lambda}(\lambda))$ . It is clear that  $\Phi \circ d = m$  where  $\Phi : \mathcal{D} \rightarrow \mathbb{N}$  is the map defined by  $\Phi(d_1, \dots, d_r) = \sum_{i=1}^r d_i$ .

Finally, we can associate to  $C \in \mathfrak{g}_\varepsilon$  a triple  $(\Lambda, m, d)$  defined as above.

**Definition 7.7.** Let  $\mathcal{J}_n$  be the set of all triples  $(\Lambda, m, d)$  such that:

- (1)  $\Lambda$  is a subset of  $\mathbb{C} \setminus \{0\}$  with  $\sharp \Lambda \leq 2n$  and  $\lambda \in \Lambda$  if and only if  $-\lambda \in \Lambda$ .
- (2)  $m : \Lambda \rightarrow \mathbb{N}^*$  satisfies  $m(\lambda) = m(-\lambda)$ , for all  $\lambda \in \Lambda$  and  $\sum_{\lambda \in \Lambda} m(\lambda) = 2n$ .
- (3)  $d : \Lambda \rightarrow \mathcal{D}$  satisfies  $d(\lambda) = d(-\lambda)$ , for all  $\lambda \in \Lambda$  and  $\Phi \circ d = m$ .

Let  $\mathcal{S}(2n)$  be the set of invertible elements in  $\mathfrak{g}_\varepsilon$  and  $\tilde{\mathcal{S}}(2n)$  be the set of  $I_\varepsilon$ -adjoint orbits of elements in  $\mathcal{S}(2n)$ . By the preceding remarks, there is a map  $i : \mathcal{S}(2n) \rightarrow \mathcal{J}_n$ . Then we have a parametrization of the set  $\tilde{\mathcal{S}}(2n)$  as follows:

**Proposition 7.8.**

*The map  $i : \mathcal{S}(2n) \rightarrow \mathcal{J}_n$  induces a bijection  $\tilde{i} : \tilde{\mathcal{S}}(2n) \rightarrow \mathcal{J}_n$ .*

*Proof.* Let  $C$  and  $C' \in \mathcal{S}(2n)$  such that  $C' = U C U^{-1}$  with  $U \in I_\varepsilon$ . Let  $S, S', N, N'$  be respectively the semisimple and nilpotent parts of  $C$  and  $C'$ . Write  $i(C) = (\Lambda, m, d)$  and  $i(C') = (\Lambda', m', d')$ . One has

$$S' + N' = U (S + N) U^{-1} = U S U^{-1} + U N U^{-1}.$$

By the unicity of Jordan decomposition,  $S' = U S U^{-1}$  and  $N' = U N U^{-1}$ . So  $\Lambda' = \Lambda$  and  $m' = m$ . Also, since  $U S = S' U$  one has  $U S(V_\lambda) = S' U(V_\lambda)$ . It implies that

$$S' (U(V_\lambda)) = \lambda U(V_\lambda).$$

That means  $U(V_\lambda) = V'_\lambda$ , for all  $\lambda \in \Lambda$ . Since  $N' = U N U^{-1}$  then  $N|_{V_\lambda}$  and  $N'|_{V'_\lambda}$  have the same Jordan decomposition, so  $d = d'$  and  $\tilde{i}$  is well defined.

To prove that  $\tilde{i}$  is onto, we start with  $\Lambda = \{\lambda_1, -\lambda_1, \dots, \lambda_k, -\lambda_k\}$ ,  $m$  and  $d$  as in Definition 7.7. Define on the canonical basis:

$$S = \text{diag}_{2n}(\overbrace{\lambda_1, \dots, \lambda_1}^{m(\lambda_1)}, \dots, \overbrace{\lambda_k, \dots, \lambda_k}^{m(\lambda_k)}, \overbrace{-\lambda_1, \dots, -\lambda_1}^{m(\lambda_1)}, \dots, \overbrace{-\lambda_k, \dots, -\lambda_k}^{m(\lambda_k)}).$$

For all  $1 \leq i \leq k$ , let  $d(\lambda_i) = (d_1(\lambda_i) \geq \dots \geq d_{r_{\lambda_i}}(\lambda_i) \geq 1)$  and define

$$N_+(\lambda_i) = \text{diag}_{d(\lambda_i)}(J_{d_1(\lambda_i)}, J_{d_2(\lambda_i)}, \dots, J_{d_{r_{\lambda_i}}(\lambda_i)})$$

on the eigenspace  $V_{\lambda_i}$  and 0 on the eigenspace  $V_{-\lambda_i}$  where  $J_d$  is the Jordan block of size  $d$ .

By Lemma 7.6,  $N(\lambda_i) = N_+(\lambda_i) - N_+^*(\lambda_i)$  is skew-symmetric on  $V_{\lambda_i} \oplus V_{-\lambda_i}$ . Finally,

$$\mathbb{C}^{2n} = \bigoplus_{1 \leq i \leq k}^{\perp} (V_{\lambda_i} \oplus V_{-\lambda_i}).$$

Define  $N \in \mathfrak{g}_{\varepsilon}$  by  $N(\sum_{i=1}^k v_i) = \sum_{i=1}^k N(\lambda_i)(v_i)$ ,  $v_i \in V_{\lambda_i} \oplus V_{-\lambda_i}$  and  $C = S + N \in \mathfrak{g}_{\varepsilon}$ . By construction,  $i(C) = (\Lambda, m, d)$ , so  $\tilde{i}$  is onto.

To prove that  $\tilde{i}$  is one-to-one, assume that  $C, C' \in \mathcal{J}(2n)$  and that  $i(C) = i(C') = (\Lambda, m, d)$ . Using the previous notation, since their respective semisimple parts  $S$  and  $S'$  have the same spectrum and same multiplicities, there exist  $U \in I_{\varepsilon}$  such that  $S' = USU^{-1}$ . For  $\lambda \in \Lambda$ , we have  $U(V_{\lambda}) = V'_{\lambda}$  for eigenspaces  $V_{\lambda}$  and  $V'_{\lambda}$  of  $S$  and  $S'$  respectively.

Also, for  $\lambda \in \Lambda$ , if  $N$  and  $N'$  are the nilpotent parts of  $C$  and  $C'$ , then  $N''(V_{\lambda}) \subset V_{\lambda}$ , with  $N'' = U^{-1}N'U$ . Since  $i(C) = i(C')$ , then  $N|_{V_{\lambda}}$  and  $N'|_{V'_{\lambda}}$  have the same Jordan type. Since  $N'' = U^{-1}N'U$ , then  $N''|_{V_{\lambda}}$  and  $N'|_{V'_{\lambda}}$  have the same Jordan type. So  $N|_{V_{\lambda}}$  and  $N''|_{V_{\lambda}}$  have the same Jordan type. Therefore, there exists  $D_+ \in \mathcal{L}(V_{\lambda})$  such that  $N''|_{V_{\lambda}} = D_+N|_{V_{\lambda}}D_+^{-1}$ . By Lemma 7.6, there exists  $D(\lambda) \in I_{\varepsilon}(V_{\lambda} \oplus V_{-\lambda})$  such that

$$N''|_{V_{\lambda} \oplus V_{-\lambda}} = D(\lambda)N|_{V_{\lambda} \oplus V_{-\lambda}}D(\lambda)^{-1}.$$

We define  $D \in I_{\varepsilon}$  by  $D|_{V_{\lambda} \oplus V_{-\lambda}} = D(\lambda)$ , for all  $\lambda \in \Lambda$ . Then  $N'' = DND^{-1}$  and  $D$  commutes with  $S$  since  $S|_{V_{\pm\lambda}}$  is scalar. Then  $S' = (UD)S(UD)^{-1}$  and  $N' = (UD)N(UD)^{-1}$  and we conclude that  $C' = (UD)C(UD)^{-1}$ .  $\square$

#### 7.4. Adjoint orbits in the general case.

Let us now classify  $I_{\varepsilon}$ -adjoint orbits of  $\mathfrak{g}_{\varepsilon}$  in the general case as follows. Let  $C$  be an element in  $\mathfrak{g}_{\varepsilon}$  and consider the Fitting decomposition of  $C$

$$\mathbb{C}^m = V_N \oplus V_I,$$

where  $V_N$  and  $V_I$  are stable by  $C$ ,  $C_N = C|_{V_N}$  is nilpotent and  $C_I = C|_{V_I}$  is invertible. Since  $C$  is skew-symmetric,  $B_{\varepsilon}(C^k(V_N), V_I) = (-1)^k B_{\varepsilon}(V_N, C^k(V_I))$  for any  $k$  then one has  $V_I = (V_N)^{\perp}$ . Also, the restrictions  $B_{\varepsilon}^N = B_{\varepsilon}|_{V_N \times V_N}$  and  $B_{\varepsilon}^I = B_{\varepsilon}|_{V_I \times V_I}$  are non-degenerate. Clearly,  $C_N \in \mathfrak{g}_{\varepsilon}(V_N)$  and  $C_I \in \mathfrak{g}_{\varepsilon}(V_I)$ . By Subsection 7.1 and Subsection 7.3,  $C_N$  is attached with a partition  $[d] \in \mathcal{P}_{\varepsilon}(n)$  and  $C_I$  corresponds to a triple  $T \in \mathcal{J}_{\ell}$  where  $n = \dim(V_N)$ ,  $2\ell = \dim(V_I)$ . Let  $\mathcal{D}(m)$  be the set of all pairs  $([d], T)$  such that  $[d] \in \mathcal{P}_{\varepsilon}(n)$  and  $T \in \mathcal{J}_{\ell}$  satisfying  $n + 2\ell = m$ . By the preceding remarks, there exists a map  $p : \mathfrak{g}_{\varepsilon} \rightarrow \mathcal{D}(m)$ . Denote by  $\mathcal{O}(\mathfrak{g}_{\varepsilon})$  the set of  $I_{\varepsilon}$ -adjoint orbits of  $\mathfrak{g}_{\varepsilon}$  then we obtain the classification of  $\mathcal{O}(\mathfrak{g}_{\varepsilon})$  as follows:

**Proposition 7.9.** *The map  $p : \mathfrak{g}_{\varepsilon} \rightarrow \mathcal{D}(m)$  induces a bijection  $\tilde{p} : \mathcal{O}(\mathfrak{g}_{\varepsilon}) \rightarrow \mathcal{D}(m)$ .*

*Proof.* Let  $C$  and  $C'$  be two elements in  $\mathfrak{g}_{\varepsilon}$ . Assume that  $C$  and  $C'$  lie in the same  $I_{\varepsilon}$ -adjoint orbit. It means that there exists an isometry  $P$  such that  $C' = PCP^{-1}$ . So  $C'^k P = P C^k$  for any  $k$  in  $\mathbb{N}$ . As a consequence,  $P(V_N) \subset V'_N$  and  $P(V_I) \subset V'_I$ .

However,  $P$  is an isometry then  $V'_N = P(V_N)$  and  $V'_I = P(V_I)$ . Therefore, one has

$$C'_N = P_N C_N P_N^{-1} \text{ and } C'_I = P_I C_I P_I^{-1},$$

where  $P_N = P : V_N \rightarrow V'_N$  and  $P_I = P : V_I \rightarrow V'_I$  are isometries. It implies that  $C_N, C'_N$  have the same partition and  $C_I, C'_I$  have the same triple. Hence, the map  $\tilde{p}$  is well defined.

For a pair  $([d], T) \in \mathcal{D}(m)$  with  $[d] \in \mathcal{P}_\varepsilon(n)$  and  $T \in \mathcal{J}_\ell$ , we set a nilpotent map  $C_N \in \mathfrak{g}_\varepsilon(V_N)$  corresponding to  $[d]$  as in Section 7.1 and an invertible map  $C_I \in \mathfrak{g}_\varepsilon(V_I)$  as in Proposition 7.8 where  $\dim(V_N) = n$  and  $\dim(V_I) = 2\ell$ . Define  $C \in \mathfrak{g}_\varepsilon$  by  $C(X_N + X_I) = C_N(X_N) + C_I(X_I)$ , for all  $X_N \in V_N, X_I \in V_I$ . By construction,  $p(C) = ([d], T)$  and  $\tilde{p}$  is onto.

To prove  $\tilde{p}$  is one-to-one, let  $C, C' \in \mathfrak{g}_\varepsilon$  such that  $p(C) = p(C') = ([d], T)$ . Keep the above notations, since  $C_N$  and  $C'_N$  have the same partition then there exists an isometry  $P_N : V_N \rightarrow V'_N$  such that  $C'_N = P_N C_N P_N^{-1}$ . Similarly  $C_I$  and  $C'_I$  have the same triple and then there exists an isometry  $P_I : V_I \rightarrow V'_I$  such that  $C'_I = P_I C_I P_I^{-1}$ . Define  $P : V \rightarrow V$  by  $P(X_N + X_I) = P_N(X_N) + P_I(X_I)$ , for all  $X_N \in V_N, X_I \in V_I$  then  $P$  is an isometry and  $C' = P C P^{-1}$ . Therefore,  $\tilde{p}$  is one-to-one.  $\square$

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MINH THANH DUONG, DEPARTMENT OF PHYSICS, UNIVERSITY OF EDUCATION OF HO CHI MINH CITY, 280 AN DUONG VUONG, HO CHI MINH CITY, VIETNAM.

ROSANE USHIROBIRA, INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UNIVERSITÉ DE BOURGOGNE, B.P. 47870, F-21078 DIJON CEDEX & NON-A INRIA LILLE - NORD EUROPE, FRANCE

*E-mail address:* thanhndmi@hcmup.edu.vn

*E-mail address:* Rosane.Ushirobira@u-bourgogne.fr